The Quadratic Oil Extraction Oligopoly

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Abstract

Each extractor has a distinct initial endowment of oil and a distinct quadratic extraction cost and faces a linear industry demand schedule. We observe in a discrete-time model with a finite number of periods that the open loop and closed loop solutions are the same if initial stocks are such that each competitor is extracting in every period in which her competitors are extracting.

- key words: oligopoly extractors, closed loop solution
- classification: D430, Q320

1 Introduction

The best hope for understanding oil extraction markets would appear to be via oligopoly theory.[1] Here we present a new quadratic revenue, quadratic extraction cost case, each extractor with her own distinct costs and initial endowments, in which open loop and closed loop competition yields the same extraction paths. The sufficiency condition for the solutions to be the same here is simply that the distinct quadratic extraction costs and endowments (the initial finite stocks) must be such that each competitor-extractor produces positive quantities in every period. We work in discrete time with a finite number of periods. Our results do not carry over to continuous time because each of many extractors will in general never be producing over the identical and finite time horizon.[2]

2 The Model

The inverted market demand schedule is $a-b[q_t^1+q_t^2]$, a and slope b positive. An extractor's current profit, given q_t^1 currently extracted, is $\pi^1(q_t^1,q_t^2)=(a-b[q_t^1+q_t^2])q_t^1-[q_t^1]^2d^1$. We work in discrete time.[3] S_t^i is firm $i\ 's$ current stock remaining. $S_{t+1}^i=S_t^i-q_t^i$. Each firm starts initially with a fixed endowment, S_0^i. We work here in the text with two extractors for

[1] Eswaran and Lewis [1985] presented two interesting discrete-time examples in which open loop and closed loop competition among extracting oligopolists yielded the same paths of extraction. For one case competitors had distinct endowments and faced a constant elasticity industry demand schedule. However, each player was required to have no extraction costs. This has recently been re-worked in continuous time by Benchakroun and Long [2005] and has been employed in an interesting exercise. In the other case, each firm had identical endowments and identical quadratic extraction costs and the industry demand schedule was linear. We are then generalizing this last example by allowing for each firm to have distinct quadratic extraction costs and distinct initial stocks.

[2] We are indebted to a referee for this observation.

[3] We follow Eswaran and Lewis [1985]. The continuous time treatment of our problem might be simpler to work out because endpoint conditions are quite restrictive. We discuss endpoint conditions below for our discrete time formulation. Levhari and Mirman [1980] is a classic early closed loop oligopoly problem in discrete time.

ease of exposition. In Appendix 2 we report on cases with more than two extractors. Each extractor has extraction cost $[q_t^i]^2 d^i$ with $d^i > 0$. Our focus is on the case of $d^1 \neq d^2$.(The case of $d^1 = d^2$ is one of the two cases reported on in Eswaran and Lewis [1985].)

For the open loop case, each extractor maximizes her present value of profits by choice of a quantity stream, $\{q_1^i, q_2^i, ..., q_{T-1}^i\}$, taking the quantity stream of each competitor as parametric. We specify initial stocks so that each competitor solves with $q_1^i + q_2^i + ... + q_{T-1}^i = S_0^i$; and these initial stocks are such that each competitor produces positive output in the same period as another competitor is producing. β is the constant discount factor, $0 < \beta < 1$, the same for each extractor. We can distinguish two cases. (1) "knife-edged" endpoints[4]: in this case the initial quantities are such that in the final period, the q_{T-1}^i $'s$ are such that marginal profit, $\frac{\partial \pi^i}{\partial q_t^i} = mr^i(q_t^i, q_t^j) - mc^i(q_t^i)$, for each firm satisfies,

$$[mr^1(q_{T-1}^1, q_{T-1}^2) - mc^1(q_{T-1}^1)] = \beta a = [mr^2(q_{T-1}^1, q_{T-1}^2) - mc^2(q_{T-1}^2)].$$

The intuition here is that $q_T^1 = q_T^2 = 0$ for this case. Working back in time, we have

$$\begin{aligned}
[mr^1(q_{T-2}^1, q_{T-2}^2) - mc^1(q_{T-2}^1)] &= \beta^2 a = [mr^2(q_{T-2}^1, q_{T-2}^2) - mc^2(q_{T-2}^2)], \\
[mr^1(q_{T-3}^1, q_{T-3}^2) - mc^1(q_{T-3}^1)] &= \beta^3 a = [mr^2(q_{T-3}^1, q_{T-3}^2) - mc^2(q_{T-3}^2)], \\
&\text{and so on, ...}
\end{aligned}$$

The above backward recursion allows us to solve explicitly for each quantity extracted. We get $\widehat{q}_{T-1}^1 = \frac{(1-\beta^1)a[b+2d^2]}{D}$, $\widehat{q}_{T-2}^1 = \frac{(1-\beta^2)a[b+2d^2]}{D}$, $\widehat{q}_{T-3}^1 = \frac{(1-\beta^3)a[b+2d^2]}{D}$, $\widehat{q}_{T-4}^1 = \frac{(1-\beta^4)a[b+2d^2]}{D}$, ... and $\widehat{q}_{T-1}^2 = \frac{(1-\beta^1)a[b+2d^1]}{D}$, $\widehat{q}_{T-2}^2 = \frac{(1-\beta^2)a[b+2d^1]}{D}$, $\widehat{q}_{T-3}^2 = \frac{(1-\beta^3)a[b+2d^1]}{D}$, $\widehat{q}_{T-4}^2 = \frac{(1-\beta^4)a[b+2d^1]}{D}$, ...

[4] The "knife-edge" terminal condition is central to continuous time dynamic optimization problems. See Gelfand and Fomin [1963, p. 60]. Lozada [1993] discusses terminal condtitions for discrete time problems and compares the "knife-edge" terminal condition with the "general" terminal condition. In brief, dynamic problems end with a very restrictive condition in the "knife-edge" case and end somewhat "ragged" in general. Our analysis focuses on the general case.

and so on, with $D = (2b + 2d^1)(2b + 2d^2) - b^2$. This knife-edged endpoint solution turns on special values for initial stocks and is too special to merit much attention but the $\widehat{q}$ expressions turn out to be useful as components of the solution quantities for the general case.

(2) "general" endpoints: initial quantities are such that in the final period, the $q^i_{T-1}\ 's$ are such that marginal profit for each firm satisfies,

$$\begin{aligned} [mr^1(q^1_{T-1}, q^2_{T-1}) - mc^1(q^1_{T-1})] &> \beta a \\ \text{and } [mr^2(q^1_{T-1}, q^2_{T-1}) - mc^2(q^2_{T-1})] &> \beta a \end{aligned}$$

and $S^i_{T-1} = q^i_{T-1}$, $i = 1, 2$. For this general case, there is a complicated backward recursion yielding the solution values for quantities extracted and this general case is the one which we focus on here.

3 Solving the Closed Loop Problem

In the closed loop case, competition among extractors is re-opened *de novo* at each consecutive period, contingent on each player taking current stock levels as the current state of the system. There is no commitment at period zero to an extraction path as there is with open loop competition. Closed loop competition requires competitive outcomes to be worked out for each period in a backward recursion or by dynamic programming arguments.

In the final period, we have

$$\begin{aligned} V^1_{T-1}(q^1_{T-1}, q^2_{T-1}) &= \pi^1(q^1_{T-1}, q^2_{T-1}) \\ \text{and } V^2_{T-1}(q^1_{T-1}, q^2_{T-1}) &= \pi^2(q^1_{T-1}, q^2_{T-1}) \\ \text{with } q^1_{T-1} &= S^1_{T-2} - q^1_{T-2} \text{ and } q^2_{T-1} = S^2_{T-2} - q^2_{T-2}. \end{aligned}$$

There are no residual stocks at the termination of extraction for any extractor. Moving one period backwards in time toward the present, we have

$$
\begin{aligned}
V^1_{T-2}(S^1_{T-2}-q^1_{T-2},S^2_{T-2}-q^2_{T-2}) &= \max_{q^1_{T-2}}\{\pi^1(q^1_{T-2},q^2_{T-2}) \\
&\quad +\beta V^1_{T-1}(S^1_{T-2}-q^1_{T-2},S^2_{T-2}-q^2_{T-2})\} \\
\text{and } V^2_{T-2}(S^1_{T-2}-q^1_{T-2},S^2_{T-2}-q^2_{T-2}) &= \max_{q^2_{T-2}}\{\pi^2(q^1_{T-2},q^2_{T-2}) \\
&\quad +\beta V^2_{T-1}(S^1_{T-2}-q^1_{T-2},S^2_{T-2}-q^2_{T-2})\}.
\end{aligned}
$$

Assuming differentiability of the V^i_{T-2}'s, the maximizations yield

$$
\begin{aligned}
mr^1(q^1_{T-2},q^2_{T-2})-mc^1(q^1_{T-2}) &= \beta[mr^1(S^1_{T-2}-q^1_{T-2},S^2_{T-2}-q^2_{T-2}) \qquad (1)\\
&\quad -mc^1(S^1_{T-2}-q^1_{T-2})], \\
mr^2(q^1_{T-2},q^2_{T-2})-mc^2(q^2_{T-2}) &= \beta[mr^2(S^1_{T-2}-q^1_{T-2},S^2_{T-2}-q^2_{T-2}) \\
&\quad -mc^2(S^2_{T-2}-q^2_{T-2})], \qquad (2)
\end{aligned}
$$

which solve to:

$$
q^i_{T-2}=\frac{1}{1+\beta}\widehat{q}^i_{T-1}+\frac{\beta}{1+\beta}S^i_{T-2} \qquad i=1,2. \tag{3}
$$

for $\widehat{q}^i_{T-1}=\frac{(1-\beta)a[b+2d^j]}{D}$. This result in (3) is central because each competitor's current extraction is being represented as independent of the other competitor's current level of stock.[5] It is as if each competitor were extracting from her own stock, independently of the other extractor.[6] The pair of equations in (2) is also fundamental because they are a template for further backward steps in the solution.

[5] This two firm, two period result was first observed as we checked some of our detailed notes on the Eswaran-Lewis research. We were surprised to get essentially Eswaran-Lewis results on the sameness of the closed loop and open loop solutions for our considerably more general specification of each firm's initial stock and extraction costs.

[6] For the case of extraction costs simply linear, as in $\delta^i q^i$, we observe that independence also prevails with $q^i_{T-2}=A^i_{T-2}+\frac{\beta}{1+\beta}S^i_{T-2}$ for $A^1_{T-2}=[\frac{1}{3b}(a-\delta^2)-\frac{2}{3b}(a-\delta^1)]\left(\frac{\beta-1}{1+\beta}\right)$ and $A^2_{T-2}=[\frac{1}{3b}(a-\delta^1)-\frac{2}{3b}(a-\delta^2)]\left(\frac{\beta-1}{1+\beta}\right)$. For the three period, two extractor case, we have $q^i_{T-3}=A^i_{T-3}+\frac{\beta^2}{(1+\beta+\beta^2)}S^i_{T-3}$ for

4 The Backward Recursion

Next we solve for q^1_{T-3} in $V^1_{T-3}(q^1_{T-3}, q^2_{T-3}) = \max_{q^1_{T-3}}\{\pi^1(q^1_{T-3}, q^2_{T-3}) + \beta V^1_{T-2}(S^1_{T-3} - q^1_{T-3}, S^2_{T-3} - q^2_{T-3})\}$ where

$$V^1_{T-2}(S^1_{T-3} - q^1_{T-3}, S^2_{T-3} - q^2_{T-3}) = \pi^1(\widetilde{q}^1_{T-2}, \widetilde{q}^2_{T-2}) + \beta\pi^1(S^1_{T-2} - \widetilde{q}^1_{T-2}, S^2_{T-2} - \widetilde{q}^2_{T-2})$$

for $\widetilde{q}^i_{T-2} = \frac{1}{1+\beta}\widehat{q}^i_{T-1} + \frac{\beta}{1+\beta}S^i_{T-2}$ and $S^i_{T-2} = S^i_{T-3} - q^i_{T-3}$ $(i = 1, 2)$. The envelope theorem yields $\frac{dV^1_{T-2}(S^1_{T-3}-q^1_{T-3}, S^2_{T-3}-q^2_{T-3})}{dq^1_{T-3}} = \frac{d\beta\pi^1(S^1_{T-2}-\widetilde{q}^1_{T-2}, S^2_{T-2}-\widetilde{q}^2_{T-2})}{dS^1_{T-2}}\frac{dS^1_{T-2}}{dq^1_{T-3}} = \frac{-d\beta\pi^1(S^1_{T-2}-\widetilde{q}^1_{T-2}, S^2_{T-2}-\widetilde{q}^2_{T-2})}{dS^1_{T-2}}$ $= -\beta[mr^1(S^1_{T-2} - \widetilde{q}^1_{T-2}, S^2_{T-2} - \widetilde{q}^2_{T-2}) - mc^1(S^1_{T-2} - \widetilde{q}^1_{T-2})]$. Hence $\max_{q^1_{T-3}}\{\pi^1(q^1_{T-3}, q^2_{T-3}) + \beta V^1_{T-2}(S^1_{T-3} - q^1_{T-3}, S^2_{T-3} - q^2_{T-3})\}$ implies that (q^1_{T-3}, q^2_{T-3}) satisfy

$$\begin{aligned}
mr^1(q^1_{T-3}, q^2_{T-3}) - mc^1(q^1_{T-3}) &= \beta^2[mr^1(S^1_{T-2} - \widetilde{q}^1_{T-2}, S^2_{T-2} - \widetilde{q}^2_{T-2}) \\
&\quad - mc^1(S^1_{T-2} - \widetilde{q}^1_{T-2})], \\
mr^2(q^1_{T-3}, q^2_{T-3}) - mc^2(q^2_{T-3}) &= \beta^2[mr^2(S^1_{T-2} - \widetilde{q}^1_{T-2}, S^2_{T-2} - \widetilde{q}^2_{T-2}) \\
&\quad - mc^2(S^2_{T-2} - \widetilde{q}^2_{T-2})].
\end{aligned}$$

Using the result in (3), this pair solves as:[7]

$$\begin{aligned}
q^i_{T-3} &= \frac{1+\beta}{1+\beta+\beta^2}\widehat{q}^i_{T-2} + \frac{\beta^2}{1+\beta+\beta^2}[S_{T-3} - \widehat{q}^i_{T-1}] \qquad i = 1, 2 \qquad (4) \\
\text{for } \widehat{q}^i_{T-2} &= \frac{(1-\beta^2)a[b + 2d^j]}{D}.
\end{aligned}$$

We again observe each current extraction by competitor i a function of the extractor's current stock. This independence property is essential to the sameness of extraction paths under

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$A^i_{T-3} = \frac{(a-2\delta^i-\delta^j)(1+\beta-2\beta^2)}{3b(1+\beta+\beta^2)}$, $i = 1, 2$.

One can proceed to the four period case and so on. Hence it follows that our central result on the sameness of open loop and closed loop solutions obtains for the case of the general quadratic cost function, distinct for each player, and each player extracting positive quantities in each period in which a rival is extracting.

[7] Details are provided in Appendix 2, where we take up the three period, three extractor case.

open loop and closed loop competition.[8] Moving one period toward the present, we again consider first order conditions (again invoking the envelope theorem) in

$$\begin{aligned} mr^1(q^1_{T-4}, q^2_{T-4}) - mc^1(q^1_{T-4}) &= \beta^3[mr^1(S^1_{T-2} - q^1_{T-2}, S^2_{T-2} - q^2_{T-2}) \\ &\quad -mc^1(S^1_{T-2} - q^1_{T-2})], \\ mr^2(q^1_{T-4}, q^2_{T-4}) - mc^2(q^2_{T-4}) &= \beta^3[mr^2(S^1_{T-2} - q^1_{T-2}, S^2_{T-2} - q^2_{T-2}) \\ &\quad -mc^2(S^2_{T-2} - q^2_{T-2})], \end{aligned}$$

and using (3) and (4) obtain

$$\begin{aligned} q^i_{T-4} &= \frac{1+\beta+\beta^2}{1+\beta+\beta^2+\beta^3}\widehat{q}^i_{T-3} + \frac{\beta^3}{1+\beta+\beta^2+\beta^3}[S^i_{T-4} - \widehat{q}^i_{T-1} - \widehat{q}^i_{T-2}] \\ \text{for } i &= 1,2 \text{ and } \widehat{q}^i_{T-3} = \frac{(1-\beta^3)a[b+2d^j]}{D}\ . \end{aligned}$$

(In Appendix 1 we fill in the details of solving for q^i_{T-4}.) We observe the independence property present when we extend the backward recursion from the future toward the present. For $T-t$ we get[9]

$$\begin{aligned} q^i_{T-t} &= \frac{1+\beta+\beta^2+\ldots+\beta^{T-t-2}}{1+\beta+\beta^2+\beta^3+\ldots+\beta^{T-t-1}}\widehat{q}^i_{T-t-1} \\ &\quad + \frac{\beta^{T-t-1}}{1+\beta+\beta^2+\beta^3+\ldots+\beta^{T-t-1}}[S^i_{T-t} - \widehat{q}^i_{T-1} - \widehat{q}^i_{T-2} - \ldots - \widehat{q}^i_{T-t-2}] \end{aligned}$$

for $i = 1, 2$.[10]

The fact that current marginal profit for each player is the discounted marginal profit for the final period implies directly that the closed loop solution is the same as the open loop

[8] See Eswaren and Lewis [1985] for details on a general approach to the sameness of open loop and closed loop solutions for extraction games.
[9] There is a check on these derivations. We know the exact expessions for the $\widehat{q}^i_{T-t}$. We can replace the corresponding expressions without hats with these with hats and verify that in each case the right hand sides for our "formulae" match the left hand sides.
[10] There is a very similar set of formulas for the case of a single monopoly extractor with quadatic extraction costs and facing a linear demand schedule.

solution. Recall our assumption that initial stocks and extraction costs are such that each player is extracting positive quantities over the same span of periods as each other player.

We summarize. For the closed loop, two player extraction game, one solves

$$
\begin{aligned}
mr^1(q^1_{T-2}, q^2_{T-2}) - mc^1(q^1_{T-2}) &= \beta[mr^1(S^1_{T-2} - \widetilde{q}^1_{T-2}, S^2_{T-2} - \widetilde{q}^2_{T-2}) \\
&\quad - mc^1(S^1_{T-2} - \widetilde{q}^1_{T-2})], \\
mr^2(q^1_{T-2}, q^2_{T-2}) - mc^2(q^2_{T-2}) &= \beta[mr^2(S^1_{T-2} - \widetilde{q}^1_{T-2}, S^2_{T-2} - \widetilde{q}^2_{T-2}) \\
&\quad - mc^2(S^2_{T-2} - \widetilde{q}^2_{T-2})],
\end{aligned}
$$

for $q^i_{T-2} = \frac{1}{1+\beta}\widehat{q}^i_{T-1} + \frac{\beta}{1+\beta}S^i_{T-2} \qquad i = 1, 2,$

$$
\begin{aligned}
mr^1(q^1_{T-3}, q^2_{T-3}) - mc^1(q^1_{T-3}) &= \beta^2[mr^1(S^1_{T-2} - \widetilde{q}^1_{T-2}, S^2_{T-2} - \widetilde{q}^2_{T-2}) \\
&\quad - mc^1(S^1_{T-2} - \widetilde{q}^1_{T-2})], \\
mr^2(q^1_{T-3}, q^2_{T-3}) - mc^2(q^2_{T-3}) &= \beta^2[mr^2(S^1_{T-2} - \widetilde{q}^1_{T-2}, S^2_{T-2} - \widetilde{q}^2_{T-2}) \\
&\quad - mc^2(S^2_{T-2} - \widetilde{q}^2_{T-2})],
\end{aligned}
$$

for $q^i_{T-3} = \frac{1+\beta}{1+\beta+\beta^2}\widehat{q}^i_{T-2} + \frac{\beta^2}{1+\beta+\beta^2}[S_{T-3} - \widehat{q}^i_{T-1}] \qquad i = 1, 2,$

$$
\begin{aligned}
mr^1(q^1_{T-4}, q^2_{T-4}) - mc^1(q^1_{T-4}) &= \beta^3[mr^1(S^1_{T-2} - \widetilde{q}^1_{T-2}, S^2_{T-2} - \widetilde{q}^2_{T-2}) \\
&\quad - mc^1(S^1_{T-2} - \widetilde{q}^1_{T-2})], \\
mr^2(q^1_{T-4}, q^2_{T-4}) - mc^2(q^2_{T-4}) &= \beta^3[mr^2(S^1_{T-2} - \widetilde{q}^1_{T-2}, S^2_{T-2} - \widetilde{q}^2_{T-2}) \\
&\quad - mc^2(S^2_{T-2} - \widetilde{q}^2_{T-2})],
\end{aligned}
$$

for $q^i_{T-4} = \frac{1+\beta+\beta^2}{1+\beta+\beta^2+\beta^3}\widehat{q}^i_{T-3} + \frac{\beta^3}{1+\beta+\beta^2+\beta^3}[S^i_{T-4} - \widehat{q}^i_{T-1} - \widehat{q}^i_{T-2}] \quad i = 1, 2,$

... back to the initial period.

Given initial stocks, one can then solve forward, using the accounting relation, $S^i_{t+1} = S^i_t - q^i_t$, to obtain $q^i_1, q^i_2, ..., q^i_{T-1}$ and $S^i_2, S^i_3, ..., S^i_{T-2}$ for $i = 1, 2$. That is, S^1_{T-2} becomes a function of initial stock $S^1(0)$ and S^2_{T-2} is a function of initial stock $S^2(0)$.

5 Illustrative Example

The question emerges: how much variation is each player's own parameters is allowed before one loses the property of each player "playing" positive quantities in every period. We select a market inverse demand curve with intercept 10 and slope 1/2. The common discount factor is 0.8 $(= \frac{1}{1+r})$. We select the number of extractors at 2 with the second extractor with stock size 10 and cost schedule $\frac{1}{2}q_2^2$. Hence extractor 2 has marginal cost with slope unity.

Extractor 1 is treated as having fraction bet of extractor $2's$ stock and fraction alf of $2's$ marginal extraction cost. In Table 1, we report our results.

Table 1

	q^1_1	q^1_2	q^1_3	q^2_1	q^2_2	q^2_3
$alf = 1.1$, $bet = 1.1$	3.6945	3.6688	3.6366	3.4862	3.3451	3.1687
$alf = 1.1$, $bet = 0.215$	1.3732	0.7672	0.0096	3.4862	3.3451	3.1687
$alf = 0.02$, $bet = 0.4695$	3.0174	1.6767	0.0009	3.2421	3.3263	3.4316
$alf = 5.0$, $bet = 1.1$	3.1573	3.6275	4.2152	3.6205	3.3554	3.0241

Our base case is in the first row. In the second row we report that we can reduce player $1's$ stock to about 0.215 of player $2's$ stock and observe that player 1 is on the verge of having no output in the final period. Hence we infer that for our choice of parameters for demand

and cost, there is ample variation possible in relative stock sizes across players while both are "still in the game" (each producing positive quantities in every period). Observe that player $2's$ quantities produced do not vary with our changes in stock size for player 1.

In the third row, we report on experiments involving costs differences across firms. When player $1's$ marginal cost has slope 0.02 of player $2's$ marginal cost AND player $1's$ stock size is about 0.4695 of player $2's$ stock size, then player 1 is on the verge of being "out of the game" (not producing a positive quantity in every period). Note that we were unable to find a positive and small value for player $1's$ marginal cost so that player 1 was "out of the game". We had to reduce player $1's$ stock size in addition to reducing the slope of her marginal cost schedule. Note that player 2 with the higher marginal cost has her quantities rising over time.

The fourth row has player 1 with a large value of her marginal extraction cost relative to player 2. Now player 1 with the high marginal cost has her quantities increasing over time.

We infer that the model we are reporting on admits, for well-behavedness, a quite wide variation in stock sizes for each player as well as in slopes of the marginal extraction functions.

6 Concluding Remarks

Our initial scrutiny of end-point conditions for the quadratic oligopoly exhaustible resource extraction problem led us to the surprising discovery of the well-behavedness of the closed loop version with each extractor with distinct quadratic extraction costs and distinct initial holdings of stock to extract. The well-behavedness extends to our central result: closed loop and open loop solutions being the same provided each extractor is doing positive extraction when each of her competitors is doing positive extractions, a seemingly weak requirement. Recall our numerical illlustrations. With distinct extractors, it has been known that when industry demand is specified as constant elasticity and each firm has no cost of extraction, open loop and closed loop problems exhibit identical extraction paths.[11] We have arrived at a new case with extractors not only possessing distinct endowments but also possessing distinct extraction costs.

[11] See Eswaran and Lewis [1985]. With each firm with say distinct extraction costs linear in current extraction, one can immediately verify that the open loop and closed loop extraction games do not have the same solutions.

Appendix 1: Calculations for obtaining q^i_{T-4} with 2 firms

Profit for firm 1 is

$$\begin{aligned}\pi^1 &= \{a - b[q^1_{T-4} + q^2_{T-4}]\}q^1_{T-4} - d^1[q^1_{T-4}]^2 \\ &+\beta\left[\{a - b[q^1_{T-3} + q^2_{T-3}]\}q^1_{T-3} - d^1[q^1_{T-3}]^2\right] \\ &+\beta^2\left[\{a - b[q^1_{T-2} + q^2_{T-2}]\}q^1_{T-2} - d^1[q^1_{T-2}]^2\right] \\ &+\beta^3\left[\{a - b[S^1_{T-2} - q^1_{T-2} + S^2_{T-2} - q^2_{T-2}]\}[S^1_{T-2} - q^1_{T-2}] - d^1[S^1_{T-2} - q^1_{T-2}]^2\right].\end{aligned}$$

There is an analogous profit statement for firm 2, with the chief difference the presence of cost parameter d^2 in place of d^1. For these profit statements, we have explicit substitutions for q^1_{T-3}, q^1_{T-2}, and q^2_{T-3} and q^2_{T-2}, expressions already obtained earlier in the backward recursion, namely

$$\begin{aligned}q^i_{T-2} &= \frac{1}{1+\beta}\widehat{q}^i_{T-1} + \frac{\beta}{1+\beta}S^i_{T-2} \qquad i = 1, 2 \\ \text{and } q^i_{T-3} &= \frac{1+\beta}{1+\beta+\beta^2}\widehat{q}^i_{T-2} + \frac{\beta^2}{1+\beta+\beta^2}[S_{T-3} - \widehat{q}^i_{T-1}] \qquad i = 1, 2.\end{aligned}$$

When we make the substitutions and solve for $\frac{\partial\pi^1}{\partial q^1_{T-4}} = 0$. Exploiting the envelope theorem, we get

$$\begin{aligned}a - [2b + 2d^1]q^1_{T-4} - bq^2_{T-4} &= \beta^3\{a - \frac{[2b + 2d^1]}{1+\beta+\beta^2}[S^1_{T-4} - q^1_{T-4} - \widehat{q}^1_{T-1} - \widehat{q}^1_{T-2}]\} \\ &\quad -\beta^3\{\frac{b}{1+\beta+\beta^2}[S^2_{T-4} - q^2_{T-4} - \widehat{q}^2_{T-1} - \widehat{q}^2_{T-2}]\}\end{aligned}$$

and for the analogous first order condition for firm 2 in

$$a-[2b+2d^2]q^2_{T-4}-bq^1_{T-4} = \beta^3\{a-\frac{[2b+2d^2]}{1+\beta+\beta^2}[S^2_{T-4}-q^2_{T-4}-\widehat{q}^2_{T-1}-\widehat{q}^2_{T-2}]\}$$
$$-\beta^3\{\frac{b}{1+\beta+\beta^2}[S^1_{T-4}-q^1_{T-4}-\widehat{q}^1_{T-1}-\widehat{q}^1_{T-2}]\}.$$

We solve for q^1_{T-4} and q^2_{T-4} with these two linear equations to get

$$q^i_{T-4}=\frac{1+\beta+\beta^2}{1+\beta+\beta^2+\beta^3}\widehat{q}^i_{T-3}+\frac{\beta^3}{1+\beta+\beta^2+\beta^3}[S^i_{T-4}-\widehat{q}^i_{T-1}-\widehat{q}^i_{T-2}] \qquad i=1,2.$$

We emphasize that each extractor's current quantity extracted is being expressed as a linear function of her own current stock alone, even though each extractor has distinct extraction costs. Below we provide detail for solving for the q^i_{T-t} $'s$ for the 3 firm case. This detail "works" as well for the two firm case above. One can readily see the "transition" from the backward recursion from period $T-t$ to period $T-t-1$.

Appendix 2: More than Two Firms

Moving backward two periods, the first order conditions for profit maximization over the two end periods for each of the three firms are

$$a-[2b+2d^1]q^1_{T-2}-bq^2_{T-2}-bq^3_{T-2} = \beta\{a-[2b+2d^1](S^1_{T-2}-q^1_{T-2})-b(S^2_{T-2}-q^2_{T-2})$$
$$-b(S^3_{T-2}-q^3_{T-2}),$$
$$a-bq^1_{T-2}-[2b+2d^2]q^2_{T-2}-bq^3_{T-2} = \beta\{a-b(S^1_{T-2}-q^1_{T-2})-[2b+2d^2](S^2_{T-2}-q^2_{T-2})$$
$$-b(S^3_{T-2}-q^3_{T-2}),$$
$$a-bq^1_{T-2}-bq^2_{T-2}-[2b+2d^3]q^3_{T-2} = \beta\{a-b(S^1_{T-2}-q^1_{T-2})-b(S^2_{T-2}-q^2_{T-2})$$
$$-[2b+2d^3](S^3_{T-2}-q^3_{T-2}).$$

This is three linear equations in q^1_{T-2}, q^2_{T-2} and q^3_{T-2} :

$$\begin{bmatrix} [2b+2d^1] & b & b \\ b & [2b+2d^2] & b \\ b & b & [2b+2d^3] \end{bmatrix} \begin{bmatrix} q^1_{T-2} \\ q^2_{T-2} \\ q^3_{T-2} \end{bmatrix}$$

$$= \frac{1}{1+\beta} \begin{bmatrix} (1-\beta)a + \beta[2b+2d^1]S^1_{T-2} + \beta b S^2_{T-2} + \beta b S^3_{T-2} \\ (1-\beta)a + \beta b S^1_{T-2} + \beta[2b+2d^2]S^2_{T-2} + \beta b S^3_{T-2} \\ (1-\beta)a + \beta b S^1_{T-2} + \beta b S^2_{T-2} + \beta[2b+2d^3]S^3_{T-2} \end{bmatrix}.$$

The solutions are

$$\begin{aligned} q^1_{T-2} &= \left[\frac{1}{1+\beta}\right] \widehat{q}^1_{T-1} + \left[\frac{\beta}{(1+\beta)}\right] S^1_{T-2} \\ q^2_{T-2} &= \left[\frac{1}{1+\beta}\right] \widehat{q}^2_{T-1} + \left[\frac{\beta}{(1+\beta)}\right] S^2_{T-2} \\ q^3_{T-2} &= \left[\frac{1}{1+\beta}\right] \widehat{q}^3_{T-1} + \left[\frac{\beta}{(1+\beta)}\right] S^3_{T-2} \end{aligned}$$

for $\widehat{q}^1_{T-1} = \left[\frac{1-\beta}{\Delta}\right] a\{[2b+2d^2][2b+2d^3] - b[2b+2d^2] - b[2b+2d^3] + b^2\}$, $\widehat{q}^2_{T-1} = \left[\frac{1-\beta}{\Delta}\right] a\{[2b+2d^1][2b+2d^3] - b[2b+2d^1] - b[2b+2d^3] + b^2\}$, $\widehat{q}^3_{T-1} = \left[\frac{1-\beta}{\Delta}\right] a\{[2b+2d^1][2b+2d^2] - b[2b+2d^1] - b[2b+2d^2] + b^2\}$, and $\Delta = \{[2b+2d^1][2b+2d^2][2b+2d^3] - b^2[2b+2d^1] - b^2[2b+2d^2] - b^2[2b+2d^3] + 2b^3\}$. These solutions or "extraction rules" have the identical form as those for the two firm case.

What we are dealing with generically is a system of the form

$$\begin{bmatrix} k^1 & b & b \\ b & k^2 & b \\ b & b & k^3 \end{bmatrix} \begin{bmatrix} q^1 \\ q^2 \\ q^3 \end{bmatrix} = \begin{bmatrix} A + k^1 S^1 + bS^2 + bS^3 \\ A + bS^1 + k^2 S^2 + bS^3 \\ A + bS^1 + bS^2 + k^3 S^3 \end{bmatrix},$$

for k^i $'s$, $b's$, $A's$ and S^i $'s$ positive scalars. The presence of the $A's$ lead to the solution for the $\widehat{q}^i$ part of our solutions above. We are however interested in when the solution q^i depends on S^i alone. This leaves us to focus our attention on the reduced system

$$\begin{bmatrix} k^1 & b & b \\ b & k^2 & b \\ b & b & k^3 \end{bmatrix} \begin{bmatrix} q^1 \\ q^2 \\ q^3 \end{bmatrix} = \begin{bmatrix} k^1S^1 + bS^2 + bS^3 \\ bS^1 + k^2S^2 + bS^3 \\ bS^1 + bS^2 + k^3S^3 \end{bmatrix}.$$

This system is fundamental to our result that q^i solves in terms of S^i alone. We verify that this 3 equation system solves with

$$q^i = \{k^1(k^2k^3 - [b]^2) - b(bk^3 - [b]^2) + b([b]^2 - bk^2)\}S^i, \quad i = 1, 2, 3.$$

We now indicate how a proof by induction on the size of our system of equations establishes that each firm's current extraction, q^i can be expressed as a function of its own current stock S^i alone. We illustrate the induction step of moving from an $(n-1)\times(n-1)$ system, for which the result is assumed true, to an $n \times n$ system. We consider now the corresponding 4×4 system in terms of 3×3 subsystems (this illustrates the key step in an induction proof).

$$\begin{bmatrix} k^1 & b & b & b \\ b & k^2 & b & b \\ b & b & k^3 & b \\ b & b & b & k^4 \end{bmatrix} \begin{bmatrix} q^1 \\ q^2 \\ q^3 \\ q^4 \end{bmatrix} = \begin{bmatrix} k^1S^1 + bS^2 + bS^3 + bS4 \\ bS^1 + k^2S^2 + bS^3 + bS^4 \\ bS^1 + bS^2 + k^3S^3 + bS^4 \\ bS^1 + bS^2 + bS^3 + k^4S^4 \end{bmatrix}.$$

The solution for q^1 for the above system can be written

$$\frac{k^1S^1}{D} \times \det\begin{bmatrix} k^2 & b & b \\ b & k^3 & b \\ b & b & k^4 \end{bmatrix}$$

$$+\frac{bS^2}{D} \times \det\begin{bmatrix} k^2 & b & b \\ b & k^3 & b \\ b & b & k^4 \end{bmatrix} - \frac{b}{D} \times \det\begin{bmatrix} bS^1 + k^2S^2 + bS^3 + bS^4 & b & b \\ bS^1 + bS^2 + k^3S^3 + bS^4 & k^3 & b \\ bS^1 + bS^2 + bS^3 + k^4S^4 & b & k^4 \end{bmatrix}$$

$$+\frac{bS^3}{D} \times \det\begin{bmatrix} k^2 & b & b \\ b & k^3 & b \\ b & b & k^4 \end{bmatrix} - \frac{b}{D} \times \det\begin{bmatrix} k^2 & bS^1 + k^2S^2 + bS^3 + bS^4 & b \\ b & bS^1 + bS^2 + k^3S^3 + bS^4 & b \\ b & bS^1 + bS^2 + bS^3 + k^4S^4 & k^4 \end{bmatrix}$$

$$+\frac{bS4}{D} \times \det\begin{bmatrix} k^2 & b & b \\ b & k^3 & b \\ b & b & k^4 \end{bmatrix} - \frac{b}{D} \times \det\begin{bmatrix} k^2 & b & bS^1 + k^2S^2 + bS^3 + bS^4 \\ b & k^3 & bS^1 + bS^2 + k^3S^3 + bS^4 \\ b & b & bS^1 + bS^2 + bS^3 + k^4S^4 \end{bmatrix}$$

$$
\begin{aligned}
= \quad & \frac{k^1 S^1}{D} \det \begin{bmatrix} k^2 & b & b \\ b & k^3 & b \\ b & b & k^4 \end{bmatrix} \\
& -\frac{b}{D} \left\{ \det \begin{bmatrix} bS^1 & b & b \\ bS^1 & k^3 & b \\ bS^1 & b & k^4 \end{bmatrix} + \det \begin{bmatrix} k^2 & bS^1 & b \\ b & bS^1 & b \\ b & bS^1 & k^4 \end{bmatrix} + \det \begin{bmatrix} k^2 & b & bS^1 \\ b & k^3 & bS^1 \\ b & b & bS^1 \end{bmatrix} \right\} \\
& +\frac{bS^2}{D} \det \begin{bmatrix} k^2 & b & b \\ b & k^3 & b \\ b & b & k^4 \end{bmatrix} - \frac{b}{D} \det \begin{bmatrix} k^2 S^2 + bS^3 + bS^4 & b & b \\ bS^2 + k^3 S^3 + bS^4 & k^3 & b \\ bS^2 + bS^3 + k^4 S^4 & b & k^4 \end{bmatrix} \\
& +\frac{bS^3}{D} \det \begin{bmatrix} k^2 & b & b \\ b & k^3 & b \\ b & b & k^4 \end{bmatrix} - \frac{b}{D} \det \begin{bmatrix} k^2 & k^2 S^2 + bS^3 + bS^4 & b \\ b & bS^2 + k^3 S^3 + bS^4 & b \\ b & bS^2 + bS^3 + k^4 S^4 & k^4 \end{bmatrix} \\
& +\frac{bS^4}{D} \det \begin{bmatrix} k^2 & b & b \\ b & k^3 & b \\ b & b & k^4 \end{bmatrix} - \frac{b}{D} \det \begin{bmatrix} k^2 & b & k^2 S^2 + bS^3 + bS^4 \\ b & k^3 & bS^2 + k^3 S^3 + bS^4 \\ b & b & bS^2 + bS^3 + k^4 S^4 \end{bmatrix}
\end{aligned}
$$

$$= \frac{k^1 S^1}{D} \det \begin{bmatrix} k^2 & b & b \\ b & k^3 & b \\ b & b & k^4 \end{bmatrix}$$

$$-\frac{b}{D}\left\{ \det \begin{bmatrix} bS^1 & b & b \\ bS^1 & k^3 & b \\ bS^1 & b & k^4 \end{bmatrix} + \det \begin{bmatrix} k^2 & bS^1 & b \\ b & bS^1 & b \\ b & bS^1 & k^4 \end{bmatrix} + \det \begin{bmatrix} k^2 & b & bS^1 \\ b & k^3 & bS^1 \\ b & b & bS^1 \end{bmatrix} \right\},$$

since the last six terms cancel each other pairwise,

for $D = \det \begin{bmatrix} k^1 & b & b & b \\ b & k^2 & b & b \\ b & b & k^3 & b \\ b & b & b & k^4 \end{bmatrix}$, leaving the solution for q^1 simply in terms of S^1. ■

The induction step is the observation that the three terms on the right hand side in the last three pairs of terms in the penultimate large expression are each the essentials for a solution for q^i in a 3×3 system. Hence we have established that if the result is true for a 2×2 sytem, and for a $(n-1) \times (n-1)$ system, it is true for an $n \times n$ system. (In fact we established it true for a 2×2 sytem, and then made use of its validity for a 3×3 system in establishing the result for a 4×4 system. We simply illustrated the key step in a complete induction proof.) We have established that for any finite number of firms, each firm's current quantity extracted can be expressed as a function of its own current stock alone. This is the key step in characterizing the closed loop solution. (Given our calculations it is obvious that the open loop solution is the same.) Besides drawing on the quadratic nature of revenue and extraction cost for each firm, the key property in inferring that the open loop and closed loop solutions are the same is that in the solutions, each firm ends up extracting over the same number of periods as each of its competitors. This requires that each firm's endowment of stock must be "right" in order that our demonstration of the sameness of the open and closed loop solutions is valid. Hence sufficient conditions for the open loop and closed loop solutions to be the same are (a) quadratic forms for revenue and extraction cost per firm and (b) "appropriate" endowments of stock for each firm at the initial date.

In general, the systems to solve in terms of periods in the backward recursion for the three firm case are in the form

$$\begin{aligned}
& mr^1(q^1_{T-t}, q^2_{T-t}, q^3_{T-t}) - mc^1(q^1_{T-t}) \\
= \; & \beta^{t-1}[mr^1(S^1_{T-2} - q^1_{T-2}, S^2_{T-2} - q^2_{T-2}, S^3_{T-2} - q^3_{T-2}) - mc^1(S^1_{T-2} - q^1_{T-2})] \\
& mr^2(q^1_{T-t}, q^2_{T-t}, q^3_{T-t}) - mc^2(q^2_{T-t}) \\
= \; & \beta^{t-1}[mr^2(S^1_{T-2} - q^1_{T-2}, S^2_{T-2} - q^2_{T-2}, S^3_{T-2} - q^3_{T-2}) - mc^2(S^2_{T-2} - q^2_{T-2})] \\
& mr^3(q^1_{T-t}, q^2_{T-t}, q^3_{T-t}) - mc^3(q^3_{T-t}) \\
= \; & \beta^{t-1}[mr^3(S^1_{T-2} - q^1_{T-2}, S^2_{T-2} - q^2_{T-2}, S^3_{T-2} - q^3_{T-2}) - mc^3(S^3_{T-2} - q^3_{T-2})].
\end{aligned}$$

The non-mechanical step is substituting for $(S^i_{T-2} - q^i_{T-2})$ each time one moves backwards in the recursion. Crucial here is the fact that the matrix algebra is essentially the same for each period in the sequence. Hence our induction proof sketched above holds for any date for n firms. For the 3 firm case, we have for $t = 3$, the system

$$\begin{bmatrix} a \\ a \\ a \end{bmatrix} - \begin{bmatrix} [2b+2d^1] & b & b \\ b & [2b+2d^2] & b \\ b & b & [2b+2d^3] \end{bmatrix} \begin{bmatrix} q^1_{T-3} \\ q^2_{T-3} \\ q^3_{T-3} \end{bmatrix}$$

$$= \beta^2 \begin{bmatrix} a - \frac{[2b+2d^1]}{1+\beta}[S^1_{T-3} - q^1_{T-3} - \widehat{q}^1_{T-1}] - \frac{b}{1+\beta}[S^2_{T-3} - q^2_{T-3} - \widehat{q}^2_{T-1}] \\ -\frac{b}{1+\beta}[S^3_{T-3} - q^3_{T-3} - \widehat{q}^3_{T-1}] \\ a - \frac{b}{1+\beta}[S^1_{T-3} - q^1_{T-3} - \widehat{q}^1_{T-1}] - \frac{[2b+2d^2]}{1+\beta}[S^2_{T-3} - q^2_{T-3} - \widehat{q}^2_{T-1}] \\ -\frac{b}{1+\beta}[S^3_{T-3} - q^3_{T-3} - \widehat{q}^3_{T-1}] \\ a - \frac{b}{1+\beta}[S^1_{T-3} - q^1_{T-3} - \widehat{q}^1_{T-1}] - \frac{b}{1+\beta}[S^2_{T-3} - q^2_{T-3} - \widehat{q}^2_{T-1}] \\ -\frac{[2b+2d^3]}{1+\beta}[S^3_{T-3} - q^3_{T-3} - \widehat{q}^3_{T-1}] \end{bmatrix}$$

leading to solutions

$$\begin{aligned} q^1_{T-3} &= \left[\frac{(1+\beta)}{1+\beta+\beta^2}\right]\widehat{q}^1_{T-2} + \left[\frac{\beta^2}{1+\beta+\beta^2}\right][S^1_{T-2} - \widehat{q}^1_{T-1}] \\ q^2_{T-3} &= \left[\frac{(1+\beta)}{1+\beta+\beta^2}\right]\widehat{q}^2_{T-2} + \left[\frac{\beta^2}{1+\beta+\beta^2}\right][S^2_{T-2} - \widehat{q}^2_{T-1}] \\ q^3_{T-3} &= \left[\frac{(1+\beta)}{1+\beta+\beta^2}\right]\widehat{q}^3_{T-2} + \left[\frac{\beta^2}{1+\beta+\beta^2}\right][S^3_{T-2} - \widehat{q}^3_{T-1}] \end{aligned}$$

For $t = 4$, we have

$$\begin{bmatrix} a \\ a \\ a \end{bmatrix} - \begin{bmatrix} [2b+2d^1] & b & b \\ b & [2b+2d^2] & b \\ b & b & [2b+2d^3] \end{bmatrix} \begin{bmatrix} q^1_{T-4} \\ q^2_{T-4} \\ q^3_{T-4} \end{bmatrix}$$

$$= \beta^3 \begin{bmatrix} a - \frac{[2b+2d^1]}{1+\beta+\beta^2}[S^1_{T-4} - q^1_{T-4} - \widehat{q}^1_{T-1} - \widehat{q}^1_{T-2}] - \frac{b}{1+\beta+\beta^2}[S^2_{T-4} - q^2_{T-4} - \widehat{q}^2_{T-1} - \widehat{q}^2_{T-2}] \\ - \frac{b}{1+\beta+\beta^2}[S^3_{T-4} - q^3_{T-4} - \widehat{q}^3_{T-1} - \widehat{q}^3_{T-2}] \\ a - \frac{b}{1+\beta+\beta^2}[S^1_{T-4} - q^1_{T-4} - \widehat{q}^1_{T-1} - \widehat{q}^1_{T-2}] - \frac{[2b+2d^2]}{1+\beta+\beta^2}[S^2_{T-4} - q^2_{T-4} - \widehat{q}^2_{T-1} - \widehat{q}^2_{T-2}] \\ - \frac{b}{1+\beta+\beta^2}[S^3_{T-4} - q^3_{T-4} - \widehat{q}^3_{T-1} - \widehat{q}^3_{T-2}] \\ a - \frac{b}{1+\beta+\beta^2}[S^1_{T-4} - q^1_{T-4} - \widehat{q}^1_{T-1} - \widehat{q}^1_{T-2}] - \frac{b}{1+\beta+\beta^2}[S^2_{T-4} - q^2_{T-4} - \widehat{q}^2_{T-1} - \widehat{q}^2_{T-2}] \\ - \frac{[2b+2d^3]}{1+\beta+\beta^2}[S^3_{T-4} - q^3_{T-4} - \widehat{q}^3_{T-1} - \widehat{q}^3_{T-2}] \end{bmatrix}$$

leading to solutions

$$\begin{aligned} q^1_{T-4} &= \left[\frac{1+\beta+\beta^2}{1+\beta+\beta^2+\beta^3}\right]\widehat{q}^1_{T-3} + \left[\frac{\beta^3}{1+\beta+\beta^2+\beta^3}\right][S^1_{T-4} - \widehat{q}^1_{T-1} - \widehat{q}^1_{T-2}] \\ q^2_{T-4} &= \left[\frac{1+\beta+\beta^2}{1+\beta+\beta^2+\beta^3}\right]\widehat{q}^2_{T-3} + \left[\frac{\beta^3}{1+\beta+\beta^2+\beta^3}\right][S^2_{T-4} - \widehat{q}^2_{T-1} - \widehat{q}^2_{T-2}] \\ q^3_{T-4} &= \left[\frac{1+\beta+\beta^2}{1+\beta+\beta^2+\beta^3}\right]\widehat{q}^3_{T-3} + \left[\frac{\beta^3}{1+\beta+\beta^2+\beta^3}\right][S^3_{T-4} - \widehat{q}^3_{T-1} - \widehat{q}^3_{T-2}] \end{aligned}$$

and so on for additional "terms" in the backward recursion. It is easy to see how the system of equations changes with each step backwards. The central result is of course that the expressions for the solved $q's$ end up as linear functions of each own stock alone for each step back in the recursion. For arbitrary date $T-t$, an induction proof would establish the validity of the "general term", given say M instead of 3 firms.

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