B Online Appendix for "Price Improvement and Execution Risk in Lit and Dark Markets"

End of Proof (Lemma 1). Now, given a stationary and symmetric equilibrium, I show that it must be the case that any investor who submits a buy order has $z_t \geq 0$, and symmetrically for sell orders. Time subscripts are dropped, as I focus on stationary equilibria. Let γ_I denote the fill rate of order type, I, and let p_I denote order type I's price impact. Further, let investor t have a valuation equal to $z_t \geq 0$. For any order type I, there is a buy and a sell option, IB, and IS, respectively. Then, for any investor z_t, a buy order of type I is preferred to a sell order of type I if and only if:

$$\gamma_{\mathsf{IB}} \times (z_t - p_{\mathsf{IB}}) \geq \gamma_{\mathsf{IS}} \times (z_t - p_{\mathsf{IS}}) \tag{76}$$

In any symmetric equilibrium, $\gamma_{\mathsf{IB}} = \gamma_{\mathsf{IS}}$, and $p_{\mathsf{IB}} = -p_{\mathsf{IS}}$. Then, (76) becomes $z_t \geq 0$, implying that no investor with $z_t \geq 0$ would prefer IS to IB.

I can now show that for any buy order type IB to be used in equilibrium, the price impact p_{IB} must be positive. To see this, suppose instead that $p_{\mathsf{IB}} < 0$. It must be then, by symmetry, that $p_{\mathsf{IS}} > 0$. Now, because p_{IB} describes the average informativeness of investors who submit orders of type IB in equilibrium, it must be the case that some investor with $z_t < 0$ submits orders of type IB. But from the previous argument, any investor with $z_t < 0$ must prefer IS to IB. A contradiction. Thus, in any equilibrium, buy orders that are used by some investors must have a positive price impact, $p_{\mathsf{IB}} > 0$, and symmetrically for sell orders.

Lastly, I show that investors who use buy orders of type I do not prefer any other sell order type J. Consider two order types, I and J. Symmetry allows us to consider a buy order of type I, and a sell order of type J, with the reverse following analogously. That investors with $z_t \geq 0$ will not use any sell order type, in equilibrium, follows from the argument above. Suppose that an investor $z_t \geq 0$ prefers a sell order of type JS to a buy order, IS. By the argument above, this investor *must* prefer order type JB to JS. Note, finally, that we have only shown that investors with $z_t \geq 0$ do not submit sell orders in equilibrium. Because $p_{\mathsf{IB}} > 0$, any investor with $z_t \in (0, p_{\mathsf{IB}})$ will prefer to abstain from trading, or prefer another buy order type J, with $p_{\mathsf{JB}} < p_{\mathsf{IB}}$. The argument for investors not using buy orders if $z_j \leq 0$ follows by symmetry. ■

Proof (Theorem 3). Throughout, I recall the conditions from (23)-(26) in the main text.

$$\Delta^{M}_{\lambda>\lambda^*} \equiv z^M - \mathsf{E}[\delta \mid z \geq z^M] - \rho \mathsf{Pr}(z \leq -z^M) \times \left(z^M - \mathsf{E}[\delta \mid z^L \leq z < z^M]\right) \tag{77}$$

$$\Delta^{l}_{\lambda>\lambda^*} \equiv \rho \mathsf{Pr}(z \leq -z^M) \times \left(z^L - \mathsf{E}[\delta \mid z^L \leq z < z^M]\right) - l \times \left(z^L - (1-2\lambda)\mathsf{E}[\delta \mid z \geq z^M]\right) \tag{78}$$

$$\Delta^{D}_{\lambda>\lambda^*} \equiv z^D - (1-2\lambda) \times \mathsf{E}[\delta \mid z \geq z^M] \tag{79}$$

$$\Delta^{L}_{\lambda>\lambda^*} \equiv (1-2\lambda)\mathsf{E}[\delta \mid z \geq z^M] - \mathsf{E}[\delta \mid z^D \leq z < z^L] \tag{80}$$

Marginal Valuation Threshold Ranking

Let equilibrium threshold values satisfy $0 \leq z^{D*} \leq z^{L*} \leq z^{M*} \leq 1$. Then, in an equilibrium where all order types are used, I show that equilibrium threshold values (indicated by $*$) must also satisfy $0 \leq z^{D*} \leq z^L_B \leq z^{L*} \leq z^M_B \leq z^{M*} \leq 1$. Consider the equilibrium conditions derived from (23)-(26):

$$z^{M*} - f(z^{M*}, 1) - \mathsf{Pr}(z \leq -z^{M*}) \times (z^{M*} - f(z^L, z^{M*})) = 0 \tag{81}$$

$$\rho \mathsf{Pr}(z \leq -z^{M*}) \times (z^{L*} - f(z^{L*}, z^{M*})) - l^* \times (z^{L*} - f(z^{D*}, z^{L*})) = 0 \tag{82}$$

$$z^{D*} - f(z^{L*}, z^{D*}) = 0 \tag{83}$$

Let $z^{M*} > z^M_B$ and $z^{L*} < z^L_B$. Then it must be true that $\mathsf{Pr}(z \leq -z^{M*}) < \rho \mathsf{Pr}(z \leq -z^M_B)$, and that $z^{L*} > \frac{\mu z^{M*}}{2-\mu} > \frac{\mu z^M_B}{2-\mu} > z^L_B$, a contradiction. Instead, suppose then that $z^{L*} > z^L_B$. Then, $\pi_L(z^{M*}) < \pi_L(z^M_B)$, implying that the z^{M*} that solves (81) must be such that $z^{M*} < z^M_B$, a contradiction. Hence, $z^{M*} \leq z^M_B$.

Then, let $z^{M*} \leq z^M_B$ and $z^{L*} < z^L_B$. Because $f(z^{L*}, z^{M*}) < f(z^L_B, z^M_B)$, it must be that $\pi_L(z^{M*}) > \pi_L(z^M_B) \Rightarrow z^{M*} > z^M_B$, a contradiction. Thus, $z^{L*} \geq z^L_B$. Finally, it must be the case that $z^{D*} \leq z^L_B$, which obtains from solving (83) for z^{D*}: $z^{D*} = \frac{\mu z^{L*}}{2-\mu} \leq \frac{\mu z^M_B}{2-\mu} = z^L_B$.

To prove that thresholds $0 \leq z^{D*} \leq z^{L*} \leq z^{M*} \leq 1$ can only form an equilibrium when $\lambda > \lambda^*$, rearrange $\mathsf{E}[\delta \mid z^D \leq z < z^L] = (1-2\lambda) \times \mathsf{E}[\delta \mid z \geq z^M]$ to isolate for λ:

$$\lambda = \frac{1 + z^{M*} - z^{L*} - z^{D*}}{2(1+z^{M*})} \geq \frac{1 - z^{D*}}{2(1+z^{M*})} \geq \frac{1 - z^L_B}{2(1+z^M_B)} = \lambda^* \tag{84}$$

which we arrive at by the fact that $z^{D*} \leq z^L_B$.

Existence and Uniqueness

For an equilibrium to exist where threshold values satisfy $0 < z^D < z^L \leq z^M < 1$, it must be true that $(1-2\lambda) \times \mathsf{E}[\delta \mid z \geq z^M] < \mathsf{E}[\delta \mid z^L \leq z < z^M]$, by Lemma 2. The argument above restricts the ranges of the threshold values to $0 \leq z^D \leq z^L_B \leq z^L \leq z^M \leq z^M_B < 1$.

Step 1: Existence and Uniqueness of $l^*(z^L)$

To show that a unique $l^* \in [0, \rho\mathsf{Pr}(z \leq -z^M)]$ exists that solves (78) for all $z^M \in [z^L_B, z^M]$ and $z^M \in [z^L, z^M_B]$, rearrange (78):

$$l^* = \frac{\rho\mathsf{Pr}(z \leq -z^M) \times (z^L - \mathsf{E}[\delta \mid z^L \leq z < z^M])}{z^L - (1-2\lambda)\mathsf{E}[\delta \mid z \geq z^M]} < \rho\mathsf{Pr}(z \leq -z^M) \tag{85}$$

Thus, a unique $l^* < \rho\mathsf{Pr}(z \leq -z^M)$ exists, as the denominator of (85) is positive when (79) is satisfied. Moreover, $(1-2\lambda) \times \mathsf{E}[\delta \mid z \geq z^M] < \mathsf{E}[\delta \mid z^L \leq z < z^M]$ implies that the inequality is strict.

Step 2: Existence and Uniqueness of $z^{D*}(z^L)$

I now show that there exists a unique $z^D \in [0, z^L]$ that solves (79) for all $z^L \in [z^L_B, z^M]$, and $z^M \in [z^L, z^M_B]$. By combining (79) and (80), we can solve to obtain $z^{D*} = \frac{\mu z^L}{2-\mu}$, which exists uniquely for all $z^L \in [z^L_B, z^M]$.

Step 3: Existence and Uniqueness of $z^{M*}(z^L)$

To show that there exists a unique $z^{M*} \in [z^L, z^M_B]$ that solves (45) for all $z^L \in [z^L_B, z^M]$, evaluate $\Delta^M_{\lambda>\lambda^*}(z^M)$ at the endpoints:

$$\Delta^M_{\lambda>\lambda^*}(z^M = z^L) = \left(z^L - \tfrac{\mu(1+z^L)}{2}\right) - \rho\tfrac{(1-z^L)}{2} \times (z^L - \mu z^L) \tag{86}$$

$$\Delta^M_{\lambda>\lambda^*}(z^M = z^M_B) = z^M_B - \tfrac{\mu(1+z^M_B)}{2} - \rho\tfrac{(1-z^M_B)}{2} \times \left(z^M_B - \tfrac{\mu(z^M_B+z^L)}{2}\right) \tag{87}$$

To see that (87) is non-negative, consider $z^L = z^L_B$. Then, (87) is zero from the proof of Theorem 1. Thus, for any $z^L > z^L_B$, it must be that $\Delta^M_{\lambda>\lambda^*}(z^M = z^M_B) > 0$, as it is increasing in z^L.

Consider Equation (86). If $z^L = z^L_B$, then $\Delta^M_{\lambda>\lambda^*} < 0$ because $\Delta^M_{\lambda>\lambda^*}(z^M = z^M_B, z^L = z^L_B) = 0$, and $\frac{\partial \Delta^M_{\lambda>\lambda^*}}{\partial z^M} > 0$. Then, coupled with the fact that (87) is non-negative, we have that there exists a $\widehat{z^M} = z^L$ such that $\Delta^M_{\lambda>\lambda^*} = 0$. Thus, for all $z^M < \widehat{z^M}$, equation (86) is negative. Therefore, $z^{M*} < \widehat{z^M}$ exists. Then, the uniqueness of $z^{M*} < \widehat{z^M}$ follows from $\Delta^M_{\lambda>\lambda^*}$ increasing in z^M:

$$\frac{\partial \Delta^M_{\lambda>\lambda^*}}{\partial z^M} = 1 - \frac{\mu}{2} + \frac{\rho}{2} \times (z^M - \mathsf{E}[\delta \mid z^L \le z < z^M]) - \rho \mathsf{Pr}(z \le -z^M) \times \left(1 - \tfrac{\mu}{2}\right) > 0 \tag{88}$$

For $z^{M*} > \widehat{z^M}$, I show that $z^{M*} = z^L$ is the unique value. Suppose that $z^{M*} = z^L$. z^{D*} and λ^* follow as in steps 2 and 3. To show that a unique l^* exists, we can combine equilibrium conditions (77) and (78), and substitute $\mathsf{E}[\delta \mid \mathsf{O} = \mathsf{DB}^*] = \frac{\mu z^L}{2-\mu}$ to achieve:

$$\Delta^M_{\lambda>\lambda^*}(z^M = z^L) = \left(z^L - \tfrac{\mu(1+z^L)}{2}\right) - l^* \times \left(z^L - \tfrac{\mu z^L}{2-\mu}\right) \tag{89}$$

By inspection, any $l^* \in [0, \rho \mathsf{Pr}(z \le -z^M)]$ that solves (89), $\forall z^L \in [\widehat{z^M}, z^M_B]$ is unique. Hence, if $z^{M*} > \widehat{z^M}$, a unique equilibrium exists where $z^{M*} = z^{L*}$. Thus, z^{M*} exists and is unique for all $z^L \in [z^L_B, z^M_B]$.

Step 4: Existence and Uniqueness of z^{L*}

To show there exists a unique $z^{L*} \in [z^L_B, z^{M*}]$ that solves (80) for all $\lambda \ge \lambda^*$, I evaluate (80) at z^{D*}:

$$\Delta^L_{\lambda>\lambda^*}(z^L) = z^L - (1-2\lambda)(1+z^{M*}) \times (1 - \tfrac{\mu}{2}) \tag{90}$$

Thus, the existence of z^{L*} depends on λ. To determine the bounds on λ, first obtain $\frac{\partial \Delta^L_{\lambda>\lambda^*}(z^L)}{\partial z^L}$:

$$\frac{\partial \Delta^L_{\lambda>\lambda^*}(z^L)}{\partial z^L} = 1 - (1-2\lambda)\left(1 - \frac{\mu}{2}\right) \times \frac{\partial z^{M*}}{\partial z^L} > 0 \tag{91}$$

which is positive by the fact that $\Delta^M_{\lambda>\lambda^*}$ is increasing in z^M and z^L: if z^M increases, then z^L must decrease for $\Delta^M_{\lambda>\lambda^*} = 0$. Hence, $\frac{\partial z^{M*}}{\partial z^L} < 0$. $\Delta^L_{\lambda>\lambda^*}(z^L)$ is increasing in z^L. Now, evaluate $z^{L*} = z^L_B$, its lower bound. Doing so implies that $z^{M*} = z^M_B$, as (77) becomes as in Theorem 1. Solving for λ, I obtain $\lambda = \frac{1}{2} - \frac{z^L_B}{(2-\mu)(1+z^M_B)} \equiv \lambda_4 < \frac{1}{2}$. Hence, $\lambda \le \lambda_4$. Moreover, at $\lambda = \lambda_4$ it must be that $l^* = 0$, as $z^{D*} - \mathsf{E}[\delta \mid z^{D*} \le z \le z^L_B] < 0$ for all $z^{D*} < z^M_B$. Now evaluate $z^L = z^{M*}$, the upper bound, and solve for λ to obtain $\lambda = \frac{2-\mu(1+z^{M*})}{2(2-\mu)(1+z^{M*})} \equiv \lambda_3 > \lambda^*$. Thus $z^{L*} = z^{M*} \in [z^L_B, z^M_B]$ at $\lambda = \lambda_3$.

Finally, to characterize z^{L*} for $\lambda \in [\lambda^*, \lambda_3)$, consider some $\tilde{\lambda} \in [\lambda^*, \lambda_3)$. Then by (91), $z^L(\tilde{\lambda}) > z^{L*}(\lambda_3) = z^{M*}(\lambda_3) > z^{M*}(\tilde{\lambda})$, $\Rightarrow\Leftarrow$. Now, let $z^{L*} = z^{M*}$. This implies that $\Delta^M_{\lambda>\lambda^*} > 0$ and $\Delta^l_{\lambda>\lambda^*} < 0$ for all $\lambda \in [\lambda^*, \lambda_3)$, and hence, any investor with valuation $z \ge z^{M*}$ prefers market orders to limit orders, and with $z < z^{M*}$ prefers dark orders to limit orders. The only check that remains is to show that z^{M*} forms an equilibrium such that $\Delta^M_{\lambda>\lambda^*} - \Delta^l_{\lambda>\lambda^*} = 0$ (i.e., investors are indifferent to market orders and dark orders at $z^{L*} = z^{M*}$). We then have the condition:

$$\Delta^M(z^{M*}) = z^{M*} - \frac{(1+z^{M*})\mu}{2} - l^*\left(z^{M*} - \frac{\mu(z^{M*}+z^{M*})}{2}\right) = 0 \tag{92}$$

which, because $z^L = z^{M*} > z^L_B$, holds for a (unique) $l^* \in (0, \rho\mathsf{Pr}(z \leq -z^{M*})$. Thus, a unique equilibrium exists for all $\lambda \in [\lambda^*, \lambda_4)$ such that $z^{L*} = z^{M*}$. ■

End of Proof (Proposition 4). Similarly, for the values under Theorem 3 where $\lambda \geq \lambda^*$, I compute W from (29). Inputting the equilibrium values for l^* and $\rho\mathsf{Pr}(\mathsf{O} = \mathsf{MS}^*)$ yields the simplification:

$$W = \frac{1-\mu}{4}\left(1 - z^{M*2} + \frac{\rho(1-z^{M*})}{2}(z^{M*2} - z^{L*2}) + l^*(z^{L*2} - z^{D*2})\right) = \frac{1-\mu}{4}\left(1 - \frac{\mu z^{M*}}{2-\mu}\right) \tag{93}$$

Hence, $W(z^{M*}) \geq W_B \iff z^{M*} \geq z^M_B$, which is true for all $\lambda \in [\lambda^*, \lambda_3)$. For any other $\lambda \in [\lambda^*, 1)$, the equilibrium is as in Theorem 1, where $W(z^{M*}) = W_B$. Finally, (72) and (93) provide that W is decreasing in z^{M*}, and thus welfare comoves negatively with the quoted spread, $\mathsf{ask} - \mathsf{bid} = \mu(1+z^{M*})$, and positively with lit market volume, $(1 - z^{M*})$. ■

End of Proof (Proposition 5). To show that price efficiency (the unconditional price impact) is less efficient in all cases, first consider $\lambda \in [\lambda_1, \lambda_2]$. I can simplify (31) by substituting equilibrium values for l^* and z^{L*} to obtain:

$$PD(\lambda \in [\lambda_1, \lambda_2]) = 1 - \mu^2 \times \left(1 - z^{M*2} + \frac{((2-\mu)z^{M*} - \mu)(z^{M*2} - z^{D*2})}{(2-\mu)z^{M*} - \mu z^{D*}} + \left(1 - \left(\frac{\mu}{2-\mu}\right)^2\right) z^{D*2}\right) \tag{94}$$

First, let z^{D*} be independent of z^{M*}. Then, (94) is increasing in z^{M*} for any fixed z^{D*}:

$$\frac{\partial PD(\lambda \in [\lambda_1, \lambda_2])}{\partial z^{M*}} = \frac{\mu^3((2-\mu)(z^{M*2} + z^{D*2}) - 2\mu z^{M*} z^{D*})(1 - z^{D*})}{(2-\mu)^2} > 0 \tag{95}$$

Thus, $PD(\lambda \in [\lambda_1, \lambda_2])$ minimizes at $z^{M*} = z^M_B \; \forall z^{D*}$. Evaluate $PD(\lambda \in [\lambda_1, \lambda_2]; z^{M*} = z^M_B) - PD_B$:

$$PD(\lambda \in [\lambda_1, \lambda_2]; z^{M*} = z^M_B) - PD_B = \frac{(z^{M\,2}_B - z^{D*2})\mu^3(4(1-\mu)(1-z^{D*}) + \mu^2(1+z^M_B) - 2\mu z^M_B)}{(2-\mu)^2(2z^M_B - \mu(z^{D*} + z^M_B))} \tag{96}$$

which is non-negative for all z^{D*} if:

$$4(1-\mu)(1-z_B^M)+\mu^2(1+z_B^M)-2\mu z_B^M \geq 0 \tag{97}$$

By the fact that $z^{D*} \leq z_B^M$. Then, evaluating (97) at z_B^M in (44), the expression is non-negative for all $(\mu,\rho) \in (0,1)^2$ by graphical inspection (see Online Appendix, Figure 7). Now, let $\lambda \in [\lambda^*, 1)$. Simplifying (31), I obtain:

$$PD(\lambda \in [\lambda^*, \lambda_4]) = 1-\mu^2 \times \left(1-z^{L*2}+\frac{\rho(1-z^{M*})}{2} \times \left(z^{L*}-\frac{\mu z^{M*}}{2-\mu}\right) z^{L*}\right) \tag{98}$$

To show that $PD(\lambda \in [\lambda^*, \lambda_4]) \geq PD_B$, first note that (98) is decreasing in z^{L*}:

$$\frac{\partial PD(\lambda \in [\lambda^*, \lambda_4])}{\partial z^{L*}} = -\frac{z^{M*}(1-z^{M*})\mu + 2(2-\mu)(1+z^{M*})z^{L*})}{2(2-\mu)} \tag{99}$$

Then, evaluating $PD(\lambda \in [\lambda^*, \lambda_4]) - PD_B$ at the lower bound of $z^{L*} = \frac{\mu z_B^M}{2-\mu}$ yields:

$$PD(\lambda \in [\lambda^*, \lambda_4]) - PD_B = \frac{\mu^4 \rho z_B^M (1-z^{M*})(z_B^M - z^{M*})}{2(2-\mu)^2} \tag{100}$$

which is non-negative for all $z^{M*} \in [z^{L*}, z_B^M]$. ■

End of Proof (Proposition 6).

Step 2. The expression for total volume is given by: $TV \equiv 2 \times (\mathsf{Pr}(\mathsf{O} = \mathsf{MB}^*) + l^* \mathsf{Pr}(\mathsf{O} = \mathsf{DB}^*))$. First, I write $TV(\lambda \in [\lambda_1, \lambda_2])$ and simplify in terms of z^{D*} and z^{M*}:

$$TV(\lambda \in [\lambda_1, \lambda_2]) = \frac{1-z^{M*}}{2} + l^* \times \frac{(z^{M*}-z^{D*})}{2} = \frac{1-z^{M*}}{2} + \frac{((2-\mu)z^{M*}-\mu)(z^{M*}-z^{D*})}{(2(2-\mu)z^{M*}-\mu z^{D*})} \tag{101}$$

Recall from the proof of Theorem 2 that $\lambda_2 < \frac{2-\mu-\mu z^{M*}}{2(2-\mu)(1+z^{M*})} \equiv \tilde{\lambda}$. Then, since (101) is decreasing in z^{D*}, and z^{D*} is decreasing in λ, it must be that $TV(\lambda \in [\lambda_1, \lambda_2]) < TV(\lambda = \tilde{\lambda})$. Hence, I can prove that $TV(\lambda \in [\lambda_1, \lambda_2])$ is lower than for some $\lambda \in [\lambda^*, \lambda_4]$ by proving that $TV(\lambda = \lambda_3) - TV(\lambda = \tilde{\lambda})$ is positive. Evaluate $z^{D*} = (1-2\lambda)(1+z^{M*}) - z^{M*}$ in (101) at $\lambda = \tilde{\lambda}$ to obtain $TV(\lambda = \tilde{\lambda}) = \frac{2-(1+z^{M*})\mu}{4}$, which is decreasing in z^{M*}. Thus, $TV(\lambda = \tilde{\lambda})$ is maximized at the lowest z^{M*}, given by $z^{M*} = z_B^M$ from (44). By simplification, $TV(\lambda = \lambda_3) - TV(\lambda = \tilde{\lambda}; z^{M*} = z_B^M) \geq 0$ if and only if:

$$\mu(2-\mu) + r_1(\mu,\rho) \geq 2\sqrt{4(1-\mu)^2\rho^2 - (16-48\mu+44\mu^2-12\mu^3)\rho + (2-\mu)^4)} \tag{102}$$

where $r_1(\mu, \rho) = \sqrt{(1-\mu)^2\rho^2 - (4 + 10\mu - 6\mu^2)\rho + (2-\mu)^2}$. Then, by graphical inspection on $(\mu, \rho) \in (0,1)^2$, (102) holds (see Online Appendix, Figure 7). Thus, $TV(\lambda = \lambda_3) \geq TV(\lambda = \tilde{\lambda}) > TV(\lambda \in [\lambda_1, \lambda_2])$. Thus, the price improvement that maximizes total volume must be in the "large price improvement" interval, which by Propositions 2-4 implies that lit market volume increases, the quoted spread narrows, and investor welfare improves compared to the benchmark equilibrium. ■

B.1 Optimality of Investor and Liquidity Provider Strategies.

In equilibrium, investors submit market orders, limit orders at competitive prices, or do not trade. An investor's deviation from one equilibrium action to another will not affect equilibrium bid and ask prices or probabilities of the future order submissions. By Lemma 2, investors cannot profitably deviate from the prescribed equilibrium actions, based on threshold decision rules, when their choices are restricted to one of the following actions: submit a market order (to buy or to sell), submit a limit order at the prescribed competitive equilibrium bid or ask price, or abstain from trading. If an investor submits a limit order at a price off-the-equilibrium path, the liquidity provider reacts to mitigate any incentive for subsequent investors to deviate from the equilibrium strategy. I detail the liquidity provider's response to off-equilibrium actions in section B.2 of the Online Appendix.

B.2 Out-of-Equilibrium Limit Orders and Beliefs

In this paper, I employ the perfect Bayesian equilibrium concept. On-the-equilibrium-path, investors submit limit orders with competitive limit prices. Off-the-equilibrium path, I require an appropriate set of beliefs to ensure that competitive limit prices strategically dominate any off-equilibrium-path deviations in limit price. Intuitively, any limit order posted at a price worse than the competitive equilibrium price is strategically dominated by the competitive price, as the professional liquidity provider reacts to the non-competitive order by undercutting it. For non-competitive limit orders that undercut the competitive price (i.e., a price inside the competitive spread), however, it is not immediate that the competitive price strategically dominates.

Perfect Bayesian equilibrium prescribes that investors and the professional liquidity provider update their beliefs by Bayes rule, whenever possible, but does not place any restrictions on the beliefs of market participants when they encounter an out-of-equilibrium action. To support competitive prices in equilibrium, I assume that if a limit buy order is posted at a price different to the competitive equilibrium bid price bid^*_{t+1}, then market participants hold the following beliefs regarding this investor's private information at period t.

If a limit buy order is posted at a price $\widehat{\text{bid}} < \text{bid}^*_{t+1}$, then market participants assume that the investor followed the equilibrium strategy, but erred when pricing the order. The professional liquidity provider then updates his expectation about δ_t to the equilibrium value and posts a buy limit order at bid^*_{t+1}. The original investor's limit order then executes with zero probability.

If a limit buy order is posted at a price $\widehat{\text{bid}} > \text{bid}^*_{t+1}$, then participants believe that this order stems from an investor with a sufficiently high valuation (e.g., $z_t = 1$) and update their expectations about δ_t to $\mathsf{E}[\delta_t \mid \widehat{\text{bid}}]$ accordingly.The new posterior expectation of V_t equals to $p_{t-1} + \mathsf{E}[\delta_t \mid \widehat{\text{bid}}]$. The professional liquidity provider is then willing to post a bid price $\text{bid}^{**}_{t+1} \leq p_{t-1} + \mathsf{E}[\delta_t \mid \widehat{\text{bid}}] + \mathsf{E}[\delta_{t+1} \mid \text{MS}_{t+1}]$. With the out-of-the-equilibrium belief of $\delta_t = 1$ and with the bid-ask spread< 1, a limit order with the new price bid^{**}_{t+1} outbids any limit buy order that yields investors positive expected profits.

The beliefs for an out-of-equilibrium sell order are symmetric. These out-of-equilibrium beliefs ensure that no investor deviates from his equilibrium strategy. I emphasize that these beliefs and actions do *not* materialize in equilibrium. Instead, they can be thought of as a "threat" to ensure that investors do not deviate from their prescribed equilibrium strategies.

B.3 Price Efficiency Numerical Examples

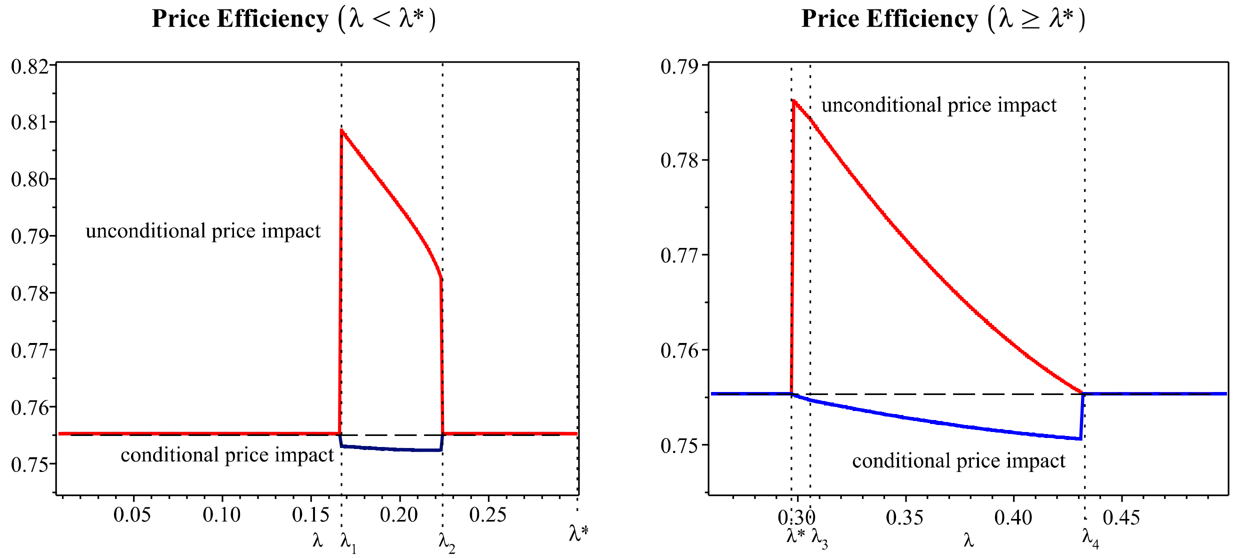

Figure 6: Price Efficiency: The panels below depict price efficiency as a function of the price improvement ($\lambda < \lambda^*$ on the left, $\lambda \geq \lambda_4$ on the right). Conditional price impact (blue line) indicates the expected price impact of an investor entering at period t conditional on a trade occurring; unconditional price impact (red line) describes the expected price impact of an investor entering the market at period t. The horizontal dashed line indicates the lit market only benchmark value; higher values than the benchmark are less efficient. Vertical dashed lines mark values for λ_1, λ_2, λ^*, λ_3 and λ_4. Parameters $\mu = 0.5$ and $\rho = 0.95$. Results for other values of μ and ρ are qualitatively similar.

B.4 Graphical Proofs

Figure 7: Graphical Proofs

The panels below depict three plots in $(\mu, \rho) \in (0,1)^2$ that serve to show that the referenced equations are above zero for all μ and ρ. From top to bottom, the figures correspond to equations (70), (97), and (102).