Price Improvement and Execution Risk in Lit and Dark Markets[*]

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Abstract

Dark pools offer price improvement over displayed quotes, but non-displayed liquidity implies execution uncertainty. Because investor limit orders also provide price improvement with execution risk, dark pools offer a natural substitute. In a model of informed trading in a market with a displayed limit order book and a dark pool that offers price improvement, higher valuation investors sort into order types with lower execution risk, generating an "immediacy hierarchy". Dark pool price improvement predicts the order in the hierarchy: a price improvement closer to (farther from) the mid-quote positions dark orders below (above) limit orders, which improves (worsens) market quality and welfare. A dark pool that is operated by the limit order book is welfare-improving, while welfare reduces with an independently-operated pool. Because active and passive order flow migrate to the dark pool where price impact occurs only post-trade, price efficiency worsens with any positive level of dark trading.

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1 Introduction

Dark trading has rapidly grown in recent years to capture a significant share of total equity volume. Rosenblatt Securities (2016) estimates that dark pools now account for roughly 17% of U.S. equity volume, up from just 4% in 2005 (Patterson, 2013). Given the increasing importance of dark venues, attention has turned to how their presence impacts market quality and investor welfare. U.S. regulators have asked whether the segmentation of order flow to dark venues provides a sufficient improvement in execution quality to offset its redirection from displayed markets that play a larger role in price discovery (SEC, 2010).

Can dark pools provide an opportunity to investors that is absent from displayed limit order books? To attract order flow, many dark venues improve on displayed quotes; the impact on execution quality is uncertain, however, as the nature of hidden liquidity implies higher risk of non-execution. Menkveld, Yueshen, and Zhu (2016) document that 11.6% of total U.S. volume occurs in dark pools that provide some level of price improvement. They hypothesize that investors route their orders according to an "immediacy pecking order": investors prioritize maximal price improvement over execution certainty if their demand for immediacy is low, but move down the pecking order to venues with lower execution risk (and smaller price improvement) for higher immediacy demands. Lit markets then sit below dark pools in the pecking order, as they offer guaranteed execution, but zero price improvement.

In the immediacy pecking order, lit markets provide execution certainty, and in this way, a dark pool offers a seemingly unique trade-off.[1] This paper contends, however, that such a trade-off exists within the lit market already, through limit orders: investors can avoid crossing the spread by improving on the current quote, thereby earning price improvement, but at the risk that their order is never filled. Because limit orders and dark orders provide a similar trade-off to market orders, it is important to discern their relative ranking in the immediacy pecking order, so that we may further understand the role of dark pools. In my analysis, I show that price improvement is a key determinant in this ranking, and moreover, that the ranking has a dichotomous impact on market quality and investor welfare.

In my model, risk-neutral investors arrive sequentially to trade at either a lit market using either a limit order or market order, or at dark pool that offers an improvement upon the prevailing displayed quote by a percentage of the bid-ask spread.[2] Investors have heterogeneous valuations that arise either from private

[1] See e.g., Hendershott and Mendelson (2000), Ye (2011), Zhu (2014), etc., for models that focus on the trade-off between displayed market orders and dark orders.

[2] For instance, prior to minimum price improvement regulation in Canada, the dark pools MatchNow and Alpha IntraSpread used trade-at rules of 20% and 10%, respectively.

information about the security, or from liquidity needs. Both the lit market and dark pool are monitored by a competitive, uninformed liquidity provider that operates as a de-facto market maker. The liquidity provider possesses a monitoring advantage towards interpreting and reacting to market data, an advantage that he uses to ensure that the limit order book is always full, and that posted quotes at the lit market are priced competitively. Consequently, market orders are guaranteed execution. Limit orders, however, are subject to execution risk, as they fill only if the subsequent investor submits the appropriate market order. Liquidity at the dark pool is hidden, and because the liquidity provider is bound by the price improvement, the liquidity provider maintains competitive quotes by choosing the intensity of liquidity provision such that the quote at the dark pool equals the (post-trade) price impact of a dark trade.

In equilibrium, an investor's trading behavior depends on the magnitude of their valuation, independent of its source (i.e., private information or liquidity needs), as well as an order's execution risk, and price impact. An important consequence of informed investors pooling with liquidity investors with the same valuation is that all investor orders—including limit orders—have price impact. Hence, the professional liquidity provider anticipates that any equilibrium limit order has an expected price impact, and will undercut the investor's limit order if this price improvement is not factored into the limit price.

Ceteris paribus, investors prefer low execution risk. In equilibrium, demand for low execution risk leads to an increase in price impact for the lowest execution risk order types, relative to others. Consequently, investors with lower valuations migrate to order types with lower relative price impact, but higher relative execution risk. The result is an "immediacy hierarchy" where investors with the highest immediacy demands sort into order types that provide the lowest execution risk (i.e., market orders), and similarly with moderate and low valuation investors. Thus, which investors use dark orders and limit orders depends on their relative position in the hierarchy. Moreover, I find that the dark pool price improvement dictates this ranking: a small price improvement (closer to the NBBO) generates lower execution risk for dark orders than displayed limit orders, whereas a large price improvement (closer to the midpoint) has the reverse effect.

The relative ranking of dark orders and displayed limit orders has important, dichotomous implications for the impact of dark pools on market quality and investor welfare. To assess the impact, I compare an environment with a displayed limit order market—or 'lit market'—and a dark pool, to a lit-market-only benchmark. A dark pool that offers a small price improvement interjects itself between market orders and limit orders in the immediacy hierarchy. Consistent with Zhu (2014), investors with moderate valuations migrate from lit market orders to the dark pool, driving the market quality results: lit market and overall trading volume falls, as market orders migrate to an order type with execution risk from one without. Lower lit market volume also increases the execution risk of limit orders. Moreover, lit market orders attract only the highest valuation investors (who are, on average, more informed), leading to a widening of the quoted spread, and an increase in the price impact of market orders.

Conversely, a dark pool with a price improvement closer to the midpoint of the spread motivates the liquidity provider to ensure higher execution risk in the dark pool relative to displayed limit orders, with the goal of attracting only (relatively uninformed) low valuation investors to the dark pool. In this case, the dark pool siphons only (low valuation) limit order submitters from the lit market, reducing their attractiveness relative to market orders. Unlike existing models with fragmented dark pools, the migration of moderate valuation investors occurs from limit orders to market orders, increasing lit market volume, as well as reducing the quoted spread and market order price impact.

I examine investor welfare through allocative efficiency in private values, similar to Bessembinder, Hao, and Zheng (2015): the expected private value realized by investors per period, discounted by the probability of a trade. I find that investor welfare improves (worsens) relative to the benchmark for a large (small) price improvement, driven by the migration of investors to (from) market orders at the lit market. This result is consistent with the findings of Hollifield, Miller, Sandås, and Slive (2006) that unfilled limit orders generate the majority of welfare loss, illustrating the impact of execution risk on welfare.

I also apply my model to short-run price efficiency. I measure price efficiency as the rate at which the market incorporates new information into the public value, and find that price efficiency declines for any dark pool price improvement. Although the addition of a new venue permits investors to sort into finer groups with respect to their valuation, thereby improving the efficiency of an investor's average *post-trade* price impact, the pre-trade opacity coupled with the execution risk of dark orders prevents some dark orders from contributing to price efficiency. The resulting impact on pre-trade price efficiency dominates, worsening price efficiency with any positive level of dark trading.

To provide insight toward optimal price improvement selection, I examine a dark pool's choice of price improvement through the lens of trade volume maximization. As venues usually charge access fees for filled orders, the price improvement level that maximizes volume acts as a proxy for profit maximization. Here, market organization informs how dark pools choose their optimal price improvement level, and subsequently the impact on market quality and welfare. If a dark pool operates as independent of the lit market, the venue will choose a price improvement level to maximize dark pool volume at the expense of lit market volume, leading to market quality worsening and welfare. If the dark pool is operated by the lit market, then the lit market will choose a welfare-enhancing price improvement level that maximizes total volume.

The dichotomous impact of price improvement level on market quality and welfare may have implications for dark trading policy. Regulators in Canada and Australia have recently sought to curb the practice of marginal price improvement by imposing a requirement that dark liquidity provide a "meaningful price improvement": the lesser of one trading increment (i.e., one cent in most markets) or the midpoint of the quoted spread.[3] More recently, the Securities and Exchange Commission (2014) has proposed a similar minimum price improvement rule on dark liquidity—the so-called "trade-at rule"—as part of a new tick size pilot program. Consistent with the empirical findings of Comerton-Forde, Malinova, and Park (2018), my paper argues that a peg to the midpoint eliminates all intermediation in dark pools. I predict the resulting welfare effect to be positive when dark trading occurs at pools operated independently of any major lit market; otherwise, the impact of a midpoint rule may be largely negative.

The European Union has also sought to curb the amount of equity trading in dark pools through a per-stock volume cap as part of MiFID II/MiFIR implemented in January of 2018. Under the cap, a venue may improve upon published "reference prices" at lit venues, but may only execute a maximum of 4% of total stock volume in any 12-month period.[4] My predictions suggest that such rules may align with a positive role for dark trading: a dark pool that operates as a subsidiary of a lit market will seek to maximize total volume by ranking dark orders at the bottom of the immediacy hierarchy, the result of which is fill rates, and a positive impact on market quality and welfare. Moreover, because dark pools that operate independently have the incentive to set a price improvement level that yields dark volume in excess of lit market volume—and thus, 50% of total volume—a cap on single-venue dark trading as a percentage of total volume may dissuade such welfare-reducing venues from forming.

1.1 Related Literature

To my knowledge, this paper is the first study the substitution effect between dark orders and displayed limit orders in the context of asymmetric information. The price improvement for execution risk trade-off between market orders and limit orders is well-studied in theoretical models of displayed limit order markets (e.g., Glosten (1994), Parlour (1998), Foucault (1999), Kaniel and Liu (2006), Rosu (2009), Brolley and Malinova

[3]See Investment Industry Regulatory Organization of Canada (2011), Australian Securities and Investments Commission (2012).
[4]See Petrescu and Wedow (2017) for a detailed treatment of MiFID II/MiFIR and its effect on dark trading.

Table 1: Theoretical Studies on Dark Trading and Market Quality

	Model					Results		
	Informed	Str. Inv. LO	Fragmented?	Hidden Orders	DP Pricing	Pr. Discovery	Market Quality	Welfare
This paper	✓	✓	✓		NMid (MM)	1/2	✓	✓
Boulatov and George (2013)	✓	1/2		✓	DLOB		✓	✓
Buti and Rindi (2013)		✓		✓	DLOB		✓	✓
Buti et al. (2017)		✓	✓		Mid		✓	✓
Degryse et al. (2009)			✓		Mid		✓	✓
Hendershott and Mendelson (2000)	✓		✓		Mid	✓	✓	✓
Iyer et al. (2018)	✓		✓		Mid		✓	✓
Moinas (2011)	✓			✓	DLOB		✓	✓
Ye, L. (2017)	✓		✓		Mid	✓		
Ye, M. (2011)	✓		✓		NBBO	✓	✓	
Zhu (2014)	✓		✓		Mid/DLOB	✓	✓	

Legend. Model properties: i) informed trading, ii) investors may submit displayed limit orders, iii) fragmented dark market, iv) hidden orders within the limit order book. Dark pool pricing is classified as: midpoint crossing networks (Mid), non-midpoint crossing (NMid), NBBO (NBBO cross), DLOB (non-displayed limit orders), and MM (market maker liquidity provision).

(2013) and others). Theoretical work on dark pools (e.g., Hendershott and Mendelson (2000), Degryse, van Achter, and Wuyts (2009), Zhu (2014), Buti, Rindi, and Werner (2017), etc.,) identifies a similar trade-off between market orders sent to lit markets and crossing networks. My paper connects the two strands of literature in an informed trading environment, to study the role of displayed limit orders in the impact of dark pools on equity markets. For ease of comparison to the existing dark trading literature, the remainder of the discussion details the model features and results of the theoretical dark trading literature in Table 1; the empirical literature is treated in Table 2.

Hendershott and Mendelson (2000) considers competition between a dealer market and a crossing network where liquidity is hidden. They find that investors with high immediacy demands (i.e., investors with short-lived information, impatient liquidity traders) prefer the dealer market. Complementary to Hendershott and Mendelson (2000), Zhu (2014) studies the impact of access to a crossing network alongside a displayed market in the presence of price risk. The author finds that price discovery improves, but lit market liquidity worsens, as low-immediacy liquidity investors concentrate at the dark pool, while informed traders (and high-immediacy liquidity traders) segment to the lit market. Moreover, in a extension, the author considers a dark pool that operates as a non-displayed limit order book that imposes a minimum price improvement–"trade-at"–rule. He finds that lesser price improvement (i.e., closer to the NBBO) increases the participation of informed traders in the dark pool.

My model extends the immediacy versus price improvement trade-off by considering investor liquidity provision at the lit market. The addition of limit orders to the investor order placement strategy set introduces the substitutability between two order types (displayed limit and dark market) that offer price improvement, at a cost of execution risk. In Hendershott and Mendelson (2000) and Zhu (2014), low-immediacy demands necessarily segment investors away from market orders, to the dark pool. In contrast, I find that depending on the relative ranking of limit orders and dark orders in an "immediacy hierarchy", the availability of a dark pool may lead investors with low immediacy demands to *migrate to* lit market orders (from displayed limit orders), thus improving liquidity.

My work also relates to a wider theoretical literature on dark trading. In an environment with symmetric information, Degryse, van Achter, and Wuyts (2009) and Buti, Rindi, and Werner (2017) examine midpoint crossing dark pools. Buti, Rindi, and Werner (2017) studies the impact of dark trading when uninformed large (institutional) and small (retail) investors choose between market orders and limit orders at the displayed limit order book, and a midpoint-crossing dark pool. I complement their work by including informed liquidity provision, a feature motivated empirically by Brogaard, Hendershott, and Riordan (2018), who find that displayed limit orders contribute to price discovery. Studies by Boulatov and George (2013), Buti and Rindi (2013), and Moinas (2011) examine the impact of hidden orders within a limit order market; as I focus on fragmented hidden liquidity, my results are complementary.

My predictions may reconcile conflicting results in the empirical literature in the liquidity effects of dark trading. Anderson, Devani, and Zhang (2015), Degryse, de Jong, and van Kervel (2015), Weaver (2014) and Comerton-Forde and Putniņš (2015) analyze Canadian, Dutch, U.S., and Australian data, respectively, to find that dark trading has a largely negative impact on liquidity; using U.S. data, Buti, Rindi, and Werner (2011) find that liquidity improves. Foley and Putniņš (2016) use Canadian data to show that the impact depends on the type of dark pool: a crossing network worsens liquidity, while a dark limit order book improves it. My work suggests that the liquidity impact of dark pools depends on the relative execution risk of dark orders and displayed limit orders: a dark pool that attracts more low immediacy investors (from

Table 2: **Empirical Studies on Dark Trading and Market Quality**

	Data		Results	
	Country	Venue Type(s)	Liquidity	Informed?
Anderson et al. (2015)	Canada	DP	↓	
Bloomfield et al. (2015)	n/a	DLOB	-	×
Buti, Rindi, and Werner (2011)	U.S.	DP	↑	
Comerton-Forde & Putniņš (2015)	Australia	DP	↓	✓
Degryse et al. (2015)	Netherlands	DP, OTC, BD	↓	
Foley and Putniņš (2016)	Canada	CN / DLOB	↓ / ↑	× / ✓
Nimalendran and Ray (2014)	U.S.	CN		✓
Ready (2014)	U.S	CN	↓	✓
Weaver (2014)	U.S.	BD	↓	

Legend. Venue types: crossing networks (CN), various dark pools (DP), Over-the-counter (OTC), Broker-Dealer/Internalizer (BD), limit order book with hidden orders (DLOB). 'Liquidity' denotes results on liquidity, where ↑, -, and ↓ indicate improvement, no change, and worsening, respectively. 'Informed?' denotes results on whether dark trades contain information, where ✓indicates an affirmative result, and × indicates a negative result.

passive orders) positively impacts liquidity, while dark pools that attract moderate-immediacy investors siphon order flow from both passive *and active orders* generate a negative impact.

A key feature of my model is that non-midpoint crosses in the dark have price impact, a feature which finds support empirically (e.g., Comerton-Forde and Putniņš (2015), Foley and Putniņš (2016), Nimalendran and Ray (2014), Ready (2014)). Consistent with my prediction that lit market orders are more informative than dark market orders, Comerton-Forde and Putniņš (2015) find that while dark trades may have price impact, the order flow that migrates to dark pools is less informed, on average.[5]

1.2 Price Improvement and Market-Making in Dark Pools

Dark pools differ in many ways: pricing mechanism, degree of opacity, degree of intermediation, exclusivity, and others. As this paper focusses on price improvement in intermediated dark pools, I focus my attention to an overview of the extent to which dark pools utilize non-midpoint pricing, as well as permit liquidity

[5]The empirical literature also looks more broadly at the role of dark trading. Degryse, Karagiannis, Tombeur, and Wuyts (2018) studies the relative usage of dark orders and hidden orders, finding that the two order types are substitutes; the preferences over which depend on prevailing market conditions. In a laboratory setting where agents can trade via hidden orders in a limit order book, Bloomfield, O'Hara, and Saar (2015) find no impact on liquidity or price efficiency.

Table 3: Dark Pool Crossing Mechanisms by Weekly Volume

Order Crossing Mechanism	Dark Pool
Midpoint Crossing (25.8%)	POSIT, SIGMA X, Instinet, KCG Matchit, MS POOL (attempts to primarily cross at midpoint)
Non-Midpoint Crossing (66.1%)	Crossfinder, UBS ATS, SuperX, Level ATS, JPM-X, Barclay's LX ATS, Instinet, Instinct X,
Other (2.5%)	MS Trajectory Cross (VWAP pricing)

provision by market makers; for detailed descriptions focussing on other classifications, see Mittal (2008), Zhu (2014), and Ye (2017).

In recent years, dark pools have significant market share in equity markets worldwide: in addition to 17% of U.S. equity volume, dark pool volume has reached 10% in Europe and Canada, and 14% in Australia (Foley and Putniņš, 2016). To classify pricing mechanisms in dark pools in the U.S. by market share, I use data on total weekly volume by trades and shares by Alternative Trading Systems (ATS) reported to the Financial Industry Regulatory Authority (FINRA). For the sample period of July 18, 2016 to May 15, 2017,[6] 38 ATSs reported trade volume data to FINRA for the entire period. There are 14 ATSs that capture $\geq 2\%$ of (average) weekly dark volume, accounting for 94.4% of total weekly dark volume. The two largest venues, UBS ATS and Crossfinder, facilitate over two-fifths. Moreover, only two of the top 14 ATS by share volume, BIDS Trading (#10) and CrossStream (#14), generate an average weekly share to trade ratio greater than 300. The average weekly share to trade ratio in the remaining 12 is roughly 200 shares, providing evidence that trade sizes in dark pools are, on average, similar to equity markets.[7]

In Figure 3, I broadly categorize each of the top 14 venues by volume using the pricing mechanism through which they match orders. Dark pools accept orders with three broad types of pricing rules: i) peg orders, that are priced in accordance with some pre-specified improvement on the NBBO (e.g., at the NBBO, the midpoint, or a % price improvement), ii) (marketable) limit orders, that enable the submitter set a limit price, whereby the order fills at the pre-specified price (or better), and; iii) immediate-or-cancel (IOC) orders, that fill only if liquidity is present at the dark pool, else they are cancelled. Many non-midpoint crossing dark pools operate as dark limit order books, but some offer non-midpoint pegged orders. As documented

[6]Weekly rolling data on total trades and total shares traded (per ATS) obtained from https://otctransparency.finra.org/.

[7]The CFA Institute (2012) notes that order and transaction sizes in today's dark pools are similar to those on lit markets. Moreover, Foley and Putniņš (2016) find that after the implementation of price improvement to midpoint in Canada, the total average (dollar) trade size remained low, at approximately $7100 CAD.

Table 4: Dark Pool Market-Making Policy

Market-Making Policy	Dark Pool
No explicit policy on market-making (8.4%)	POSIT, Instinet, MS Trajectory Cross (VWAP pricing)
Allows liquidity by market-makers (86%)	Crossfinder, UBS ATS, SuperX, Level ATS, JPM-X, KCG Matchit, Barclay's LX ATS, Instinet, Instinct X, SIGMA X

in the study by Foley and Putniņš (2016), prior to the minimum price improvement rule (MPIR) in Canada, Alpha Intraspread and MatchNow offered non-midpoint peg orders at a fixed price improvement of 10% and 20%, respectively. Dark pools which permit price improvement away from the midpoint account for two-thirds of average weekly volume, consistent with the findings of Menkveld, Yueshen, and Zhu (2016). In this way, studying the impact of non-midpoint execution in dark pools can shed light on an increasingly important segment of dark trading.

In addition to variations in pricing mechanisms across dark pools, modern dark venues also attract liquidity providers from various participant classes. While traditional crossing networks operate as off-exchange matching pools for investors[8], many modern dark pools advertise (and in some cases solicit) liquidity provision by market makers. Table 4 delineates the top 14 U.S ATS (by share volume) by whether the venue explicitly advertises that they permit/solicit order flow from liquidity providers. The percentage weights noted in the left column reflect the total average weekly volume by category.

Comerton-Forde, Malinova, and Park (2018) provide evidence that market makers use dark pools. Using Canadian data, they identify and study the two largest dark pools. They document that a venue classified as 'Market Ad' sources 87.5% of its liquidity from market makers, while 'Market D' sources its liquidity from a wide variety of investors, but still generates 18% of total liquidity from market makers. Market Ad and Market D fix price improvement to 10% and 20%, respectively. Both markets also permit midpoint crossing. Given that dark pool market makers provide a meaningful source of liquidity for investors seeking price improvement, my model seeks to provide theoretical guidance as to how dark liquidity price improvement may affect market maker liquidity provision—and thus the extent of intermediation—in dark pools.

[8] Mittal (2008) provides a description of dark trading in the early 2000s.

2 Model

Security. A single risky security with an unknown fundamental value, V, follows a random walk. An innovation to the fundamental value, $\delta_t \sim U[-1,1]$, occurs at each period t. I denote $V_t = \sum_{\tau \leq t} \delta_\tau$.

Market Organization. The market consists of a displayed limit order book, and a (dark) liquidity pool, where quotes are hidden. Both markets source liquidity from a professional liquidity provider. The displayed limit order book also accepts market orders and limit orders from other market participants. The dark pool accepts only market orders from other participants. Limit orders posted to both venues live for one period, after which any unfilled orders are cancelled. At period t, the best prices at which to buy and sell at the displayed limit order market are denoted as ask_t and bid_t, respectively.

The dark pool accepts liquidity from the liquidity provider at a pre-specified price. The venue sets the price based on an exogenous price improvement on the prevailing displayed quote, $\lambda \in [0,1]$, measured as a fraction of the difference between the best quote (i.e., the quoted spread) from the lit market, $\mathsf{ask}_t - \mathsf{bid}_t$ at period t. For example, a sell limit order at the dark pool posts at $\mathsf{ask}_t - \lambda \times (\mathsf{ask}_t - \mathsf{bid}_t)$. Prices at the dark pool are denoted by the superscript 'Dark'.

Investors. In the tradition of Glosten and Milgrom (1985), a single investor randomly arrives at the market from a continuum of risk-neutral investors at the beginning of each period, t. With probability $\mu \in (0,1)$ the investor is privately informed of the period t innovation δ_t.[9] Otherwise, an uninformed investor arrives, and is endowed with liquidity needs, y_t, uniformly distributed on $[-1,1]$.[10] Informed investors have zero liquidity needs (i.e., $y_t = 0$).

Professional Liquidity Provider. There is a single, competitive professional liquidity provider that participates in both markets by submitting only limit orders. He is risk-neutral, uninformed, and has no liquidity needs. Similar to Brolley and Malinova (2013), I assume that the liquidity provider has a superior monitoring ability to other market participants, permitting him to react to any orders submitted by investors before the arrival of the next investor. Hence, within any period t, the liquidity provider updates his orders in response to market information (i.e. changes to the public history) before the arrival of the next investor.

The liquidity provider maintains a "full" displayed limit order book at any period t by posting limit buy

[9] Numerical simulations show that the results are qualitatively robust to smooth densities of the innovation δ_t, where the innovation density first-order stochastically dominates the density for the private values (i.e., as the average valuations increase, so does the price impact of the order).

[10] Assuming that some investors have liquidity needs (or private valuations, as in Parlour (1998)) is common practice in the literature on trading with asymmetric information, to avoid the no-trade result of Milgrom and Stokey (1982).

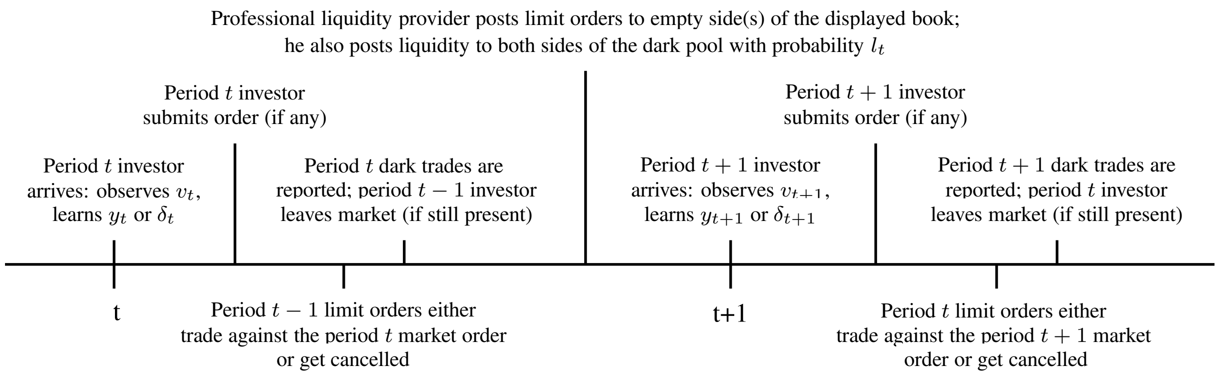

Figure 1: **Entry and Order Submission Timeline.** This figure illustrates the timing of events upon the arrival of an investor at an arbitrary period, t, until their departure from the market. y_t represents the private valuation of the period t investor (if uninformed) and δ_t is the innovation to the security's fundamental value in period t.

(sell) orders priced at $\mathsf{bid}_t^{\mathrm{LP}}$ ($\mathsf{ask}_t^{\mathrm{LP}}$) to any empty side of the book at period t. The professional liquidity provider also competitively supplies liquidity to the dark pool at each period t. Because prices for dark liquidity are dictated by the price improvement, he implements his competitive liquidity provision strategy through the intensity l_t with which he posts liquidity to both sides of the dark pool: taking investors strategies as given, the liquidity provider provides maximal liquidity ($l_t = 1$) if expected profit is positive, no liquidity ($l_t = 0$) if it is negative, and $l_t \in (0, 1)$ if it is zero.

Actions, Information, and Timing. Similar to Foucault (1999), I assume that the security trades throughout a "trading day" where the trading process ends after period t with probability $(1 - \rho) > 0$, at which point the payoff to the asset is realized. Before the start of any period t, the liquidity provider submits competitive limit orders to both sides of the lit market, and to the dark pool according to his competitive liquidity provision strategy. At the beginning of period t, an investor arrives to the market to trade. Upon arriving at the market in period t and only then, an investor may submit an order for a single unit (round lot) of the security to the displayed limit order book using a market order or a limit order (with a pre-specified limit price), or send a market order to the dark pool. I denote the choice set at period t as $\mathsf{O}_t = \left\{\mathsf{MB}_t, \mathsf{MS}_t, \mathsf{LB}_t(\mathsf{bid}_{t+1}^{\mathrm{inv}}), \mathsf{LS}_t(\mathsf{ask}_{t+1}^{\mathrm{inv}}), \mathsf{DB}_t, \mathsf{DS}_t, \mathsf{NT}_t\right\}$, which denotes market buy and sell orders, limit buy and sell orders at their respective limit prices, dark buy and sell orders, and abstaining from trade, respectively. Limit orders, if filled, execute at $t + 1$. The superscript 'inv' denotes limit prices set by investors. An investor leaves the market forever upon the execution or cancellation of their order.

All market participants observe the complete transaction history on both markets, as well as quotes and cancellations on the lit market. I denote this history up to (but not including) period t as H_t. I denote a participant's information set by I_t. Moreover, I denote the public expectation of V_t at period t as v_t. The structure of the model is common knowledge to all participants. Figure 1 illustrates the timing of the model.

Investor Payoffs. An investor's payoff to an order type is the difference between their valuation (their private value y_t, plus their expectation of the security's value) and the price, discounted by the execution probability. Investors who abstain from trade receive a payoff of zero. The payoffs to buy order types are:

$$\pi_{t,\text{inv}}^{\mathsf{MB}} = y_t + \mathsf{E}[V_t \mid I_t] - \mathsf{ask}_t \tag{1}$$

$$\pi_{t,\text{inv}}^{\mathsf{LB}} = \rho \cdot \mathsf{Pr}(\mathsf{O}_{t+1} = \mathsf{MS}_{t+1} \mid I_t, \mathsf{bid}_{t+1}^{\text{inv}}) \times \big(y_t + \mathsf{E}[V_{t+1} \mid I_t, \text{fill at } \mathsf{bid}_{t+1}^{\text{inv}}] - \mathsf{bid}_{t+1}^{\text{inv}}\big) \tag{2}$$

$$\pi_{t,\text{inv}}^{\mathsf{DB}} = l_t \times \big(y_t + \mathsf{E}[V_t \mid I_t] - \mathsf{ask}_t^{\text{Dark}}\big) \tag{3}$$

where $I_t = \{\delta_t, H_t\}$ if the investor is informed, and H_t otherwise; $\mathsf{Pr}(\mathsf{MS}_{t+1} \mid I_t, \mathsf{bid}_{t+1}^{\text{inv}})$ is the execution probability for an investor limit buy order priced at $\mathsf{bid}_{t+1}^{\text{inv}}$; Sell order payoffs are analogously defined.

Professional Liquidity Provider Payoffs. After observing the period t investor's action, the professional liquidity provider submits limit orders to the lit market, and may post orders to the dark pool. The superscript 'LP' denotes limit prices set by the liquidity provider. The payoff to a displayed buy limit order at price $\mathsf{bid}_{t+1}^{\text{LP}}$ is:

$$\pi_{t,\text{LP}}^{\mathsf{LB}} = \rho \cdot \mathsf{Pr}(\mathsf{O}_{t+1} = \mathsf{MS}_{t+1} \mid \mathsf{O}_t, I_t, \mathsf{bid}_{t+1}^{\text{LP}}) \times \big(\mathsf{E}[V_{t+1} \mid \mathsf{O}_t, I_t, \text{fill at } \mathsf{bid}_{t+1}^{\text{LP}}] - \mathsf{bid}_{t+1}^{\text{LP}}\big) \tag{4}$$

Similarly, the liquidity provider's payoff to a dark limit buy order is:

$$\pi_{t,\text{LP}}^{\mathsf{DL}} = l_t \times \mathsf{Pr}(\mathsf{O}_{t+1} = \mathsf{DS}_{t+1} \mid I_t) \times (\mathsf{E}[V_{t+1} \mid \mathsf{O}_t, H_t, \text{fill at } \mathsf{bid}_{t+1}^{\text{Dark}}] - \mathsf{ask}_{t+1}^{\text{Dark}}) \tag{5}$$

where $\mathsf{Pr}(\mathsf{O}_{t+1} = \mathsf{DS}_{t+1} \mid I_t)$ represents the probability that the period $t+1$ investor submits a sell order to the dark pool, conditional on the liquidity provider's information at t. Sell order payoffs are analogous.

3 Equilibrium

In what follows, I outline general properties for an equilibrium where a displayed limit order book operates alongside a dark pool. For equilibrium characterization, I analyze first the case where the dark pool is simply a transparent dealer market that offers only displayed quotes at a price improvement rate λ, to better distinguish the role of fragmentation from that of venue opacity on the strategies of market participants.

An equilibrium of this game consists of: i) a quoting strategy by the liquidity provider for both the displayed limit order book, and the dark pool, ii) an order placement strategy by informed and liquidity investors, and iii) for investors submitting a limit order, a limit order pricing strategy. I search for a stationary perfect Bayesian equilibrium in which investors use all order types, where possible. Moreover, I assume that investor strategies are symmetric across buyers and sellers: investors who choose the same order type (e.g. market or limit order) have private information or private valuations that are identical in *absolute value*.[11]

With respect to limit prices, I focus on equilibria where the bid and ask prices at the displayed limit order book in period t are competitive with respect to available public information at period t. Moreover, I focus attention to pooling equilibria with respect to investor limit prices: in equilibrium, investors who submit limit orders use the same limit order pricing strategy. Summarizing these properties, I define a candidate equilibrium below.

Definition 1 (Candidate Equilibrium) *A candidate equilibrium of the game described in Section 2 is a weak perfect Bayesian Nash equilibrium in which all order types are used (where possible) that satisfies:*

(1) *Stationarity: investor's strategies are independent of history, H_t.*

(2) *Symmetry: investor valuations (i.e., the sum of private information and private valuation) that are identical in absolute value use the same order type (market, limit, dark order).*

(3) *Pooling in investor limit prices: all investors who submit limit buy (sell) orders price their limit orders at the same* $\mathsf{bid}_{t+1}^{inv}(\mathsf{ask}_{t+1}^{inv})$.

In general, exposition is given from the perspective of a buyer, as the symmetric nature of the equilibria yields analogous intuition for seller decisions.

3.1 Limit Order Pricing

The professional liquidity provider supplies limit orders to both markets such that they earn zero expected profits within each market. The liquidity provider sets his quotes at the limit order market just prior to period t taking into account their information set $I_t = \{H_t, \mathsf{O}_t\}$, plus the information they can infer from order execution. I denote bid and ask prices at the limit order market by bid_t^* and ask_t^*, respectively, where

[11] While the environment is not itself stationary, given that the innovation from any period never becomes fully public knowledge, investor strategies behave as if they are stationary: because investors are risk-neutral, the expectation of the fundamental value at any period t given the public history H_t is equal to the period t midquote at the lit market, v_t.

$*$ indicates "in equilibrium". These prices are given by:

$$\mathsf{bid}_t^{\mathrm{LP}*} = \mathsf{E}[V_t \mid H_t, \mathsf{O}_t^* = \mathsf{MS}_t^*] = v_t + \mathsf{E}[\delta_t \mid \mathsf{O}_t^* = \mathsf{MS}_t^*] \tag{6}$$

$$\mathsf{ask}_t^{\mathrm{LP}*} = \mathsf{E}[V_t \mid H_t, \mathsf{O}_t^* = \mathsf{MB}_t^*] = v_t + \mathsf{E}[\delta_t \mid \mathsf{O}_t^* = \mathsf{MB}_t^*] \tag{7}$$

The limit prices at period t decompose into the public prior v_t (the mid-quote), plus the adverse selection component (the expectation over δ_t).

An investor who chooses to submit a limit order in period t must also set a limit price. The choice of limit price then serves as a signal about the informed nature of the submitter to other market participants. In an equilibrium with pooling in investor limit prices, an investor who chooses to submit a limit order in period t prices their limit order at the same limit price as all other investors who submit an order of that type. Hence, the limit order must incorporate a price improvement equal to the expected price impact of an equilibrium limit order submitter. That is, a limit buy order posted by an investor at t must include the expected price impact $\mathsf{E}[\delta_t \mid \mathsf{O}_t^* = \mathsf{LB}_t^*]$, such that $\mathsf{bid}_{t+1}^{\mathrm{inv}*} = \mathsf{bid}_{t+1}^*$:

$$\mathsf{bid}_{t+1}^{\mathrm{inv}*} = \mathsf{E}[V_{t+1} \mid H_t, \mathsf{O}_{t+1}^* = \mathsf{MS}_{t+1}^*, \mathsf{O}_t^* = \mathsf{LB}_t^*] = v_t + \mathsf{E}[\delta_t \mid \mathsf{O}_t^* = \mathsf{LB}_t^*] + \mathsf{E}[\delta_{t+1} \mid \mathsf{O}_{t+1}^* = \mathsf{MS}_{t+1}^*] \tag{8}$$

I ensure that investors submitting limit orders do not deviate from this pooling strategy by imposing appropriate off-equilibrium path beliefs on the liquidity provider with regards to his interpretation of the signals sent by off-equilibrium limit prices. I detail these beliefs fully in subsection A.5; a summary is provided in the following example. Suppose the investor instead submits a limit buy order with a worse price improvement. Because the liquidity provider immediately reacts to changes in the public history, he subsequently undercuts the investors limit order to the competitive price. However, if the investor supplies a better than competitive price—perhaps to improve execution probability—I assume that the liquidity provider holds the beliefs that the investor is informed, and has observed $\delta_t = 1$. The liquidity provider then posts a limit order to incorporate his beliefs about the prevailing price, undercutting the investor's limit order. Hence, investors who submit limit orders in equilibrium, do so at the competitive limit price. As a result, these beliefs are strictly off-the-equilibrium path.

At the dark pool, limit prices are prescribed by (a) the displayed quote, and, (b) the price improvement,

λ. These prices, denoted $\mathsf{bid}_t^{\text{Dark}*}$ and $\mathsf{ask}_t^{\text{Dark}*}$, are defined by:

$$\mathsf{bid}_t^{\text{Dark}*} = \mathsf{bid}_t^* + \lambda \times (\mathsf{ask}_t^* - \mathsf{bid}_t^*) = v_t + (1 - 2\lambda)\mathsf{E}[\delta_t \mid \mathsf{O}_t^* = \mathsf{MS}_t^*] \tag{9}$$

$$\mathsf{ask}_t^{\text{Dark}*} = \mathsf{ask}_t^* - \lambda \times (\mathsf{ask}_t^* - \mathsf{bid}_t^*) = v_t + (1 - 2\lambda)\mathsf{E}[\delta_t \mid \mathsf{O}_t^* = \mathsf{MB}_t^*] \tag{10}$$

As I search for an equilibrium in which buyer and seller decision rules are symmetric (e.g., for market orders, $\mathsf{E}[\delta_t \mid \mathsf{O}_t = \mathsf{MB}_t^*] = -\mathsf{E}[\delta_t \mid \mathsf{O}_t = \mathsf{MS}_t^*]$), applying this assumption to equations (6)-(7) admits the second equality in (9)-(10).

An important implication of investor limit orders having price impact is that investors may elect to avoid this price impact by submitting an order to the dark pool to obtain price improvement there. If we focus on price alone, $\mathsf{bid}_t^{\text{inv}*} < \mathsf{bid}_t^{\text{Dark}*}$ due to the price improvement, but because investors compare trading at the dark pool in period t with placing a limit order for execution at period $t+1$, the price impact of limit orders may make the effective price they receive *worse* than they would receive at dark pool at period t. That investor limit orders have price impact is supported empirically by Brogaard, Hendershott, and Riordan (2018), where they find that both high-frequency and non-high frequency limit orders contribute to price discovery (though primarily the former).

3.2 Order Placement Strategies

3.2.1 Investor Decision Rules

An investor will submit an order (to either market) only if, conditional on their information *and* on the submission of the order, their expected profits are non-negative. Moreover, an investor chooses the order type (if any) that maximizes their expected profits. Since investors who submit limit orders are assumed to have an identical pricing strategy, all standing limit buy orders in period t post at the same quote, regardless of the participant placing the order: $\mathsf{bid}_t^{\text{inv}*} = \mathsf{bid}_t^{\text{LP}*} = \mathsf{bid}_t^*$; similarly sell limit orders post at ask_t^*.

An investor chooses their order based on their valuation for the security, which consists of available public information H_t, plus: (a) their private valuation y_t (if uninformed), or; (b) their knowledge of δ_t, (if informed). Because their valuation enters the payoff function identically, I can summarize investor decisions in terms their valuation regardless of type. I denote the private component of a period t investor's valuation by, $z_t = y_t + \mathsf{E}[\delta_t \mid I_t]$. Because y_t and δ_t are symmetrically distributed on the interval $[-1, 1]$, so is z_t, as either $y_t = 0$ (if informed) or $\mathsf{E}[\delta_t \mid I_t] = 0$ (if uninformed). Then, an investor's valuation can be written as

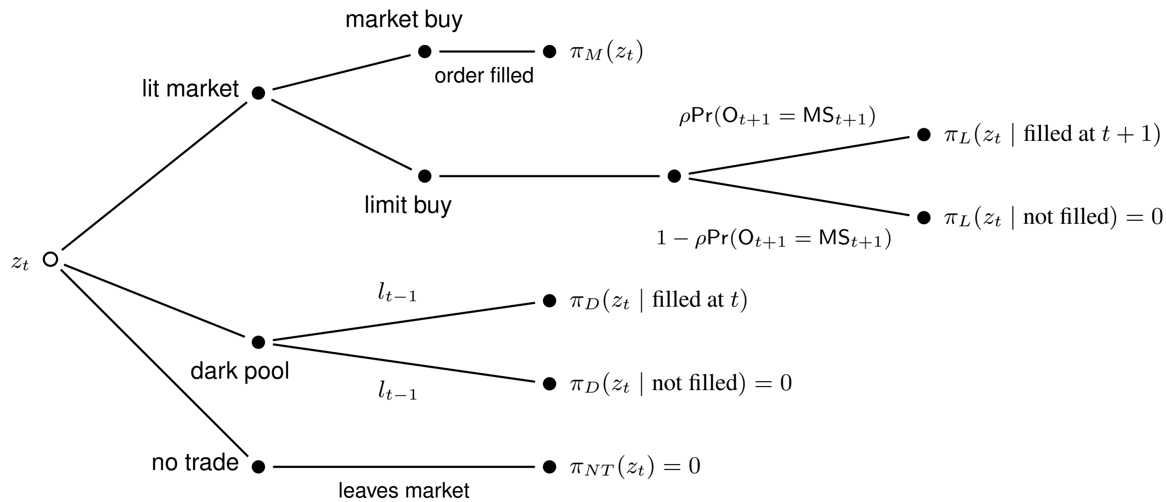

Figure 2: **Investor Decision Tree.** Given their valuation, z_t, the period t investor chooses to submit an order to the lit market, dark pool, or abstain from trading. If the investor sends an order to the lit market, the investor chooses either a market buy, or a limit buy (with the appropriate competitive price). A market order is filled automatically in period t, while a limit order is filled in period $t+1$ with probability $\mathsf{Pr}(\mathsf{MS}_{t+1})$. An order sent to the dark pool, is filled with probability l^*_{t-1}.

the private component of their valuation plus the public prior:

$$\text{investor valuation}_t = y_t + \mathsf{E}[V_t \mid I_t, H_t] = v_t + y_t + \mathsf{E}[\delta_t \mid I_t] = v_t + z_t \tag{11}$$

Then, using (11), and pricing expressions (6)-(10), we can rewrite investor payoff functions in (1)-(3) as:

$$\pi^{\mathsf{MB}}_{t,\text{inv}} = z_t - \mathsf{E}[\delta_t \mid \mathsf{O}^*_t = \mathsf{MB}^*_t] \tag{12}$$

$$\pi^{\mathsf{LB}}_{t,\text{inv}} = \rho \cdot \mathsf{Pr}(\mathsf{O}_{t+1} = \mathsf{MS}_{t+1}) \times (z_t - \mathsf{E}[\delta_t \mid \mathsf{O}^*_t = \mathsf{LB}^*_t]) \tag{13}$$

$$\pi^{\mathsf{DB}}_{t,\text{inv}} = l_t \times (z_t - (1-2\lambda)\mathsf{E}[\delta_t \mid \mathsf{O}^*_t = \mathsf{MB}^*_t]) \tag{14}$$

An important fact from (12)-(14) is that they are independent of history, H_t, and of the public expectation of the fundamental value, v_t. Hence, the model setup is internally consistent in the sense that the assumption of a stationary equilibrium does not preclude investors from maximizing their payoffs at any period t. Moreover, following the response of the professional liquidity provider to off-equilibrium path limit orders, an investor's strategy in period t is unaffected by any deviation from the equilibrium path by an investor at any prior period. Lastly, as the analysis that follows is independent of v_t, I refer to z_t as the period t investor valuation for ease of exposition, as "private valuation" is reserved for y_t. I illustrate the decision tree of an investor with valuation z_t in Figure 2.

3.2.2 Liquidity Provision at the Dark Pool

The dark pool accepts limit orders from the liquidity provider at a prescribed set of limit prices. As such, the liquidity provider chooses only the intensity with which he provides liquidity. Because the liquidity provider acts competitively within each market, he chooses his liquidity provision intensity such that he earns zero profits at the dark pool, on average. Thus, he chooses his intensity such that the price that he receives (pays) for a sell (buy) limit order posted at period $t-1$ equals the public value of the security at period $t-1$, plus the adverse selection (price impact) from filling an investor's dark order at period t. Inputting the dark pool pricing equation (10) into the liquidity provider payoff equation (4), the zero profit condition for buy orders posted to the dark pool must then satisfy:

$$\begin{aligned} & \mathsf{E}[V_{t+1} \mid \mathsf{O}_{t+1}, H_t, \text{fill at } \mathsf{ask}^{\text{Dark}}_{t+1}] - \mathsf{ask}^{\text{Dark}}_{t+1} = 0 \qquad (15) \\ \iff & (v_{t+1} + \mathsf{E}[\delta_{t+1} \mid \mathsf{O}^*_{t+1} = \mathsf{DB}^*_{t+1}]) - (v_{t+1} + (1-2\lambda) \times \mathsf{E}[\delta_{t+1} \mid \mathsf{O}^*_{t+1} = \mathsf{MB}^*_{t+1}]) = 0 \end{aligned}$$

The motivation for modelling liquidity providers as responders to a pre-set price improvement by altering the rate at which they provide liquidity can be seen in the empirical work of Comerton-Forde, Malinova, and Park (2018), where they document dark market order fill rates of two Canadian dark pools that offer price improvements of 10% and 20% on the NBBO to be 36.4% and 4%, respectively. The authors find that following a regulatory mandate to increase the minimum price improvement level to one tick (or 50% for orders with one-to-two tick spreads), dark order fill rates fell on both markets, with a larger decline on the venue with more intermediation.

Following a trade in the dark, I assume dark trades have post-trade price impact, which is facilitated through the updating of displayed quotes. In reality, dark transactions are publicly revealed (posted to the "ticker-tape") upon execution, through the Trade Reporting Facility (TRF). Because the price at which the trade occurs is away from the midpoint (for all $\lambda \neq 1/2$), the liquidity provider infers the direction of the trade, and uses this information to update their limit orders on the lit market.[12] In this way, dark trades have post-trade price impact. In what follows, I use the term 'price impact' to refer to post-trade price impact for lit market trades and dark trades, and 'pre-trade price impact' for investor limit orders.

[12]In a lit market with more than one (identical) liquidity provider, the model would require that all liquidity providers to infer the direction of the dark trade from the ticker tape and update their quotes *simultaneously*, to abstract from the sniping scenario detailed in Budish, Cramton, and Shim (2015).

3.3 Equilibrium Characterization and Existence

3.3.1 Execution Probability and Price Impact

An investor's payoff functions to each order type given by (1)-(3) reflect a key trade-off between execution probability and order price: given their valuation, an investor's profit is increasing in execution probability, and decreasing in price. If we simplify any payoff function for an order type O_t by inputting the expression for the investor's valuation (11), and the equilibrium quoting strategies for the respective order types, we find that for any period t order type O_t, the broadly-defined payoff function is given by:

$$\pi_t^O = \text{execution probability } \times (z_t - \text{price impact}_t) \tag{16}$$

For market and dark orders, (16) follows from direct substitution. For limit orders, an investor's valuation is in terms of V_{t+1}, which accounts for expected adverse selection to a filled order (i.e., expectation of δ_{t+1} conditional on their order being filled). Because the adverse selection component is priced into bid^*_{t+1} and ask_{t+1}, the component cancels, yielding an expression in the form of (16).

Thus, all else equal, an investor seeks immediacy via high execution probability. However, to earn a positive profit, an investor's valuation must exceed the price impact of their order. The following lemma provides that in any symmetric, stationary equilibrium, an order will (weakly) move the mid-quote in the direction of the investor's valuation, by the price impact (the average informativeness) of the order; this occurs with certainty if an order is sent to the limit order book, but only upon execution of a dark order.

Lemma 1 (Order Submission and Price Impact) *In any symmetric, stationary equilibrium in which investors use all order types (where possible), investors do not sell if $z_t \geq 0$ and do not buy if $z_t \leq 0$. Moreover, the price impact of a buy (sell) order in period t is positive (negative).*

If an investor's valuation exceeds the price impact of an order type, then their expected profit is increasing in the order's fill rate. Hence, investors with valuations most different from the public value have the most to gain from a higher fill rate, but consequently, are the most informed (on average), and hence contribute to a higher price impact. I conjecture that these incentives admit an equilibrium in threshold strategies where those with the most extreme valuations opt for orders with the highest fill rate, but in turn, contribute the highest price impact, such that investors with a valuation below some threshold prefer to accept a smaller execution probability, in exchange for reduced price impact. I define a threshold strategy below, and prove this conjecture in the following lemma.

Definition 2 (Threshold Strategy) *Let I and J be two order types. A threshold strategy is defined as a set of valuation thresholds z^I and z^J, such that for $z^I < z^J$:*

(1) *an investor with valuation $z_t \geq z^J$ chooses order type J;*

(2) *an investor with valuation $z^I \leq z_t < z^J$ chooses order type I;*

(3) *an investor with valuation $z_t < z^I$ chooses neither.*

Lemma 2 (Threshold Strategy and Execution Probability) *In any symmetric, stationary equilibrium in which investors use all order types (where possible), investors use a threshold strategy, as described in Definition 2. Moreover, for two order types I and J that are used in equilibrium, $z^J > z^I$ if and only if the execution risk of order type J is lower than that of I.*

Lemma 2 produces a result similar to the prediction of Hollifield, Miller, and Sandås (2004), that investors with more extreme valuations use order types with higher fill rates. They test this prediction empirically, as a monotonicity restriction on investors' order submission strategies, and find support when the choices of buy and sell orders are considered separately. Taken together, Lemmas 1 and 2 summarize the optimal order placement decision: the higher an investor's valuation, the larger the price impact that they are willing to absorb, in exchange for better execution.

Corollary 1 (Execution Probability and Price Impact) *In any symmetric, stationary equilibrium where investors use all order types, if an order type I has lower execution risk than order J, then I must also have a greater (absolute) price impact.*

The intuition of Corollary 1 is similar to the pecking order hypothesis of dark order types, coined by Menkveld, Yueshen, and Zhu (2016). They generate a pecking-order result in a model with symmetric information: investors with high private valuations submit market orders against quotes on the lit market, where they are guaranteed execution, but contribute a high price impact; investors with lower private valuations opt to send orders to a dark pool to receive a price improvement, at the expense of higher execution uncertainty. Here, I show that the result is preserved in a model with asymmetric information and investor liquidity provision: an investor weighs an order's execution likelihood against its price impact.

3.3.2 A Lit Market

In what follows, I focus on stationary equilibria, and thus drop all t subscripts. To illustrate the impact of a dark pool operating alongside a lit market, I first characterize a benchmark equilibrium to use as a basis of comparison. Consider the case where only the lit market is available (as in Brolley and Malinova (2013)). Investors choose either market orders or limit orders, or abstain from trade. As in Definition 2, I look for threshold values z^M and $z^L < z^M$ such that investors with valuations z: i) above z^M use market buy orders, ii) between z^L and z^M use limit buy orders, and; iii) between $-z^L$ and z^L abstain from trade. Sell orders are symmetric to buy orders. I obtain the following indifference conditions obtain from buy order payoff functions (1)-2), limit order pricing equations, (6)-(7), and Lemma 2, which I use to solve for z^L and z^M.

$$\pi^{\mathsf{MB}}_{\text{inv}}(z^M) = \pi^{\mathsf{LB}}_{\text{inv}}(z^M) \iff z^M = \frac{\mathsf{E}[\delta \mid z \geq z^M] - \rho \cdot \mathsf{Pr}(z \leq -z^M)\mathsf{E}[\delta \mid z^L \leq z < z^M]}{1 - \rho \cdot \mathsf{Pr}(z \leq -z^M)} \tag{17}$$

$$\pi^{\mathsf{LB}}_{\text{inv}}(z^L) = \pi^{\mathsf{NT}}_{\text{inv}}(z^L) \iff z^L = \frac{\mu z^M}{2 - \mu} \tag{18}$$

Equations (17)-(18) jointly characterize valuations z^M and z^L. Solving these equations, I obtain the following existence and uniqueness theorem, where the subscript ‘B’ denotes lit-only equilibrium values.

Theorem 1 (Equilibrium: Single Lit Market) *Let $(\mu, \rho) \in (0,1)^2$. If investors may only use the lit market, then there exist unique values $z^M \equiv z^M_B$ and $z^L \equiv z^L_B$ such that $0 < z^L_B < z^M_B < 1$, and limit prices* bid^* *and* ask^* *that satisfy (6)-(7), that constitute an equilibrium in threshold strategies (as described in Definitions 1-2). An investor who arrives at period t with valuation z:*

(i) *submits a market buy order if $z \geq z^M_B$;*

(ii) *submits a limit buy order at* bid^* *if $z^L_B \leq z < z^M_B$, and;*

(iii) *abstains from trading if $0 \leq z < z^L_B$.*

Investor sell decisions are symmetric to buy decisions.

Figure 3 illustrates how an investor with valuation z behaves under Theorem 1: investors with valuations further from the public value choose order types with lower execution risk.

To study the impact of fragmentation alone, consider the addition of a liquidity pool that offers displayed quotes at a price improvement λ—effectively a (discount) dealer market—alongside the limit order book. Because the liquidity provider plays a strategy in equilibrium that earns non-negative profits, it is always the case that $l_t = 0$ forms an equilibrium, as the liquidity provider has no incentive to provide additional liquidity given an investors’ strategy not to use the pool, and an investor will not submit an order to a pool they observe as empty. Hence, we have the following corollary.

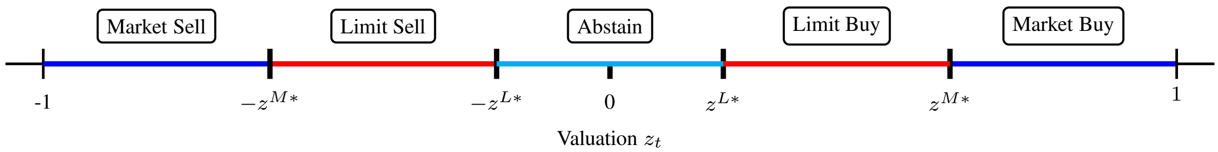

Figure 3: Lit Market Equilibrium: In the equilibrium described by Theorem 1, an investor with valuation $z_t \in [-1, 1]$ optimally chooses their order type, O_t based on the region their valuation falls into, as illustrated by the figure above.

Corollary 2 *Let a displayed liquidity pool that offers a price improvement $\lambda \in (0, 1)$ operate alongside a displayed limit order market. For all $\lambda \in (0, 1)$, there exists an equilibrium in which the liquidity provider supplies no liquidity to the pool ($l^* = 0$), and limit prices and investor strategies are as in Theorem 1.*

Now, consider the liquidity provider strategy $l^* > 0$. To a period t investor, their strategy is conditional on the state of liquidity at the pool that they observe at period t: either full or empty. Thus, irrespective of the actual strategy played by the liquidity provider, an investor always chooses between a full limit order book and either an empty or full displayed liquidity pool. In equilibrium, the displayed liquidity pool's price improvement on the competitive price at the limit order book implies that conditional on choosing to submit an order, an investor always prefers to submit an order to the (non-empty) liquidity pool. This leads to no investors using the limit order book, in equilibrium, where the following proposition obtains.

Proposition 1 (Displayed Liquidity Pool) *Let a displayed liquidity pool operate alongside a displayed limit order market. If the liquidity pool offers a price improvement $\lambda \in (0, 1)$, there does not exist an equilibrium in threshold strategies (as described in Definitions 1-2) where displayed limit prices* bid^*, ask^* *satisfy (6)-(7), $l^* > 0$, and lit market volume is positive ($z^M < 1$).*

Proposition 1 provides similar intuition to Glosten (1994), in that the electronic limit order book is robust to competition from (displayed) dealer markets. Here we see that any price improvement offered by the liquidity pool is either met by a lack of liquidity provision, or the liquidity pool attracts all active order flow from the limit order market—it offers no execution risk at a better price—leading to a break down of the reference price against which the liquidity pool determines its price improvement.[13] Hence, a discount dealer market alone cannot generate an environment where these two venues co-exist.

[13] A similar argument for $l^* = 1$ applies to the case where quotes at the liquidity pool are hidden, as investors would, in equilibrium, believe that the pool is full, and limit order market volume would thus be zero.

3.3.3 A Lit Market and a Dark Pool

Given that fragmentation alone does not segment order flow to a liquidity pool, I consider the impact of a liquidity pool with non-displayed quotes. With hidden liquidity, and thus the addition of execution risk, can a dark pool offer a price improvement λ that leads to an equilibrium with positive dark volume? First, consider $\lambda \geq 1/2$ (i.e., dark liquidity is priced equal to or better than the public value v). Since Lemma 1 implies that an investor's dark order has a (post-trade) price impact, the liquidity provider would expect to earn a loss for any $l > 0$. Hence, the liquidity provider uniquely chooses $l^* = 0$, for all $\lambda \in [1/2, 1)$, as in Theorem 1. Indeed, this equilibrium exists more generally for all $\lambda \in (0, 1)$, following from Corollary 2, with the modification that investors will not submit an order to a dark pool which they *believe* to be empty.

Now consider $\lambda \in (0, 1/2)$. To analyze how investors respond to a given price improvement, we must first understand the relative execution risk profiles across all order types. Given an investor's valuation z_t (i.e., their preference for immediacy), Lemma 2 provides that high-valuation investors prefer market orders, so that they bear no execution risk. What remains is to compare the execution risk of a dark order at price improvement $\lambda \in (0, 1/2)$, against the execution risk of an investor limit order, $\rho \mathsf{Pr}(\mathsf{O} = \mathsf{MS})$.

There are two candidate rankings of dark orders and limit orders in the immediacy pecking order, characterized by their relative execution risk. To aide with equilibrium characterization, I define the following price improvement value, $\lambda^* = \frac{1 - z_B^L}{2(1 + z_B^M)}$. The value λ^* defines the equivalent price improvement offered by investor limit orders in the benchmark equilibrium: $(1 - 2\lambda^*)\mathsf{E}[\delta \mid z \geq z_B^M] = \mathsf{E}[\delta \mid z_B^L \leq z < z_B^M]$. I use this price improvement as a marker to position the price improvement regions for under which the two candidate rankings of dark orders and limit orders admit an equilibrium.

First, conjecture that $l > \rho \mathsf{Pr}(\mathsf{O} = \mathsf{MS})$. Lemma 2 implies that investors employ a monotonic threshold strategy in their valuation z_t whereby the preference for immediacy is increasing in z_t, and thus the order type preference ranking is: i) market order, ii) dark order, iii) limit order, iv) abstain from trading. Hence, I look for a dark liquidity provision intensity such that $l > \rho \mathsf{Pr}(\mathsf{O} = \mathsf{MS}) = \rho \mathsf{Pr}(z \leq -z^M)$, and threshold values z^M, z^L, and z^D such that $0 \leq z^L \leq z^D \leq z^M \leq 1$. The preference ranking implies that the solution to the investor's problem must solve the following indifference conditions, where an investor with valuation: i) z^M is indifferent to submitting a market order or a dark order, $\pi_{\text{inv}}^{\mathsf{MB}}(z^M) = \pi_{\text{inv}}^{\mathsf{DB}}(z^M)$, ii) z^D is indifferent to submitting a dark order or a limit order, $\pi_{\text{inv}}^{\mathsf{DB}}(z^D) = \pi_{\text{inv}}^{\mathsf{LB}}(z^D)$, and; iii) z^L is indifferent to submitting a limit order, or not trading, $\pi_{\text{inv}}^{\mathsf{LB}}(z^L) = \pi_{\text{inv}}^{\mathsf{NT}}(z^L)$. Sell orders are symmetric to buy orders. Moreover, the liquidity provider solves the zero-profit condition for posting dark liquidity in (15).

Using the aforementioned indifference and zero-profit conditions, and the displayed and non-displayed equilibrium prices, (6)-(7) and (9)-(10), I solve for thresholds $z^L(z^M)$, $z^D(z^M)$, and liquidity provision intensity $l(z^M)$ in terms of z^M and λ, and provide an implicit expression for z^M.

$$z^D(z^M) = (1-2\lambda)(1+z^M) - z^M \tag{19}$$

$$z^L(z^M) = \frac{\mu}{2-\mu}((1-2\lambda)(1+z^M) - z^M) \tag{20}$$

$$l(z^M) = \frac{2z^M - \mu(1+z^M)}{2z^M - (1-2\lambda)\mu(1+z^M)} \tag{21}$$

$$z^M : \left(l^*(z^M) - \rho(1-z^M)\left(\frac{1-\mu}{2-\mu}\right)\right) z^{D*}(z^M) - \frac{(1-2\lambda)\mu(1+z^M)}{2} = 0 \tag{22}$$

The solutions to equations (19)-(22) then yield the following existence theorem.

Theorem 2 (Small Price Improvement) *Let $(\mu, \rho) \in (0,1)^2$, λ_1 and λ_2 satisfy $0 < \lambda_1 < \lambda_2 < \lambda^*$, and let dark pool quotes be hidden. Then, there exist unique values l^*, z^{L*}, z^{D*} and z^{M*} that solve (19)-(22), and limit prices bid^* and ask^* satisfying (6)-(7), that constitute an equilibrium in threshold strategies (as described in Definitions 1 and 2). Moreover, these values satisfy:*

(i) $\quad 0 < z^{L*} < z_B^L < z^{D*} < z_B^M < z^{M*} = 1$ *if and only if* $\lambda = \lambda_1$

(ii) $\; 0 < z^{L*} < z_B^L < z^{D*} < z_B^M < z^{M*} < 1$ *if and only if* $\lambda \in (\lambda_1, \lambda_2]$

Similarly, conjecture that $l < \rho\mathsf{Pr}(\mathsf{O} = \mathsf{MS})$. Hence, the order type preference ranking becomes: i) market order, ii) limit order, iii) dark order, and iv) abstain from trading. Thus, I look for equilibrium values of dark liquidity provision intensity such that $l < \rho\mathsf{Pr}(\mathsf{O} = \mathsf{MS}) = \rho\mathsf{Pr}(z \leq -z^M)$, and thresholds z^M, z^L, and z^D such that $0 \leq z^D \leq z^L \leq z^M \leq 1$. These equilibrium values are then characterized by the solutions to the liquidity provider's zero-profit condition, and the following indifference conditions: i) $\pi^{\mathsf{MB}}_{\text{inv}}(z^M) = \pi^{\mathsf{LB}}_{\text{inv}}(z^M)$, ii) $\pi^{\mathsf{LB}}_{\text{inv}}(z^L) = \pi^{\mathsf{DB}}_{\text{inv}}(z^L)$, and; iii) $\pi^{\mathsf{DB}}_{\text{inv}}(z^D) = \pi^{\mathsf{NT}}_{\text{inv}}(z^D)$. Solving the aforementioned conditions, I obtain expressions for thresholds $z^L(z^M)$, $z^D(z^M)$, and liquidity provision intensity $l(z^M)$ in terms of z^M

and λ, and an implicit expression for z^M.

$$z^L(z^M) = (1-2\lambda)(1+z^M)\left(1-\frac{\mu}{2}\right) \tag{23}$$

$$z^D(z^M) = \frac{(1-2\lambda)(1+z^M)\mu}{2} \tag{24}$$

$$l(z^M) = \frac{\rho(1-z^M)(2-\mu)\left((2-\mu)(1-2\lambda)(1+z^M)\left(1-\frac{\mu}{2}\right)-\mu z^M\right)}{8(1-\mu)(1-2\lambda)(1+z^M)(1-\frac{\mu}{2})} \tag{25}$$

$$z^M : z^M - \mathsf{E}[\delta \mid z \geq z^M] - \rho \cdot \mathsf{Pr}(z \leq -z^M)\left(z^M - (1-2\lambda)\mathsf{E}[\delta \mid z \geq z^M]\right) = 0 \tag{26}$$

The solutions to equations (23)-(26) then yield the following existence theorem.

Theorem 3 (Large Price Improvement) *Let $(\mu,\rho) \in (0,1)^2$, λ_3 and λ_4 satisfy $\lambda^* \leq \lambda_3 < \lambda_4 < 1/2$, and let dark pool quotes be hidden. Then, there exist unique values l^*, z^{L*}, z^{D*}, and z^{M*} that solve (23)-(26), and limit prices* bid^* *and* ask^* *satisfying (6)-(7), that constitute an equilibrium in threshold strategies (as described in Definitions 1 and 2). Moreover, these values satisfy:*

(i) $\quad 0 < z^{D*} \leq z_B^L < z^{L*} = z^{M*} \leq z_B^M < 1$ *if and only if* $\lambda \in [\lambda^*, \lambda_3]$

(ii) $\; 0 < z^{D*} < z_B^L \leq z^{L*} < z^{M*} \leq z_B^M < 1$ *if and only if* $\lambda \in (\lambda_3, \lambda_4]$

Similar to Figure 3, Figure 4 illustrates the behavior of an investor with valuation z under Theorems 2 and 3. When dark trading occurs in equilibrium, Theorems 2 and 3 demonstrate that price improvement informs the relative execution risk of dark orders (via Lemma 2), and thus their relative ranking in the "immediacy hierarchy": price improvement closer to the displayed quote (i.e., small price improvement) attracts moderate valuation investors, while an improvement closer to the mid-quote (i.e., large price improvement) attracts low valuation investors. If the price improvement is small, liquidity providers compensate with lower execution risk, relative to limit orders. From Corollary 1, the resulting dark orders also have greater price impact than limit orders, but the price impact is post-trade only. A dark pool that offers a large price improvement, however, provides a substantial discount over posted prices. To ensure that the professional liquidity provider breaks even, he disincentivizes moderate-valuation investors from sending orders to the dark by providing liquidity less frequently, thus increasing execution risk at the dark pool beyond that of limit orders. Corollary 1 then implies that dark trades have lower price impact than limit orders.

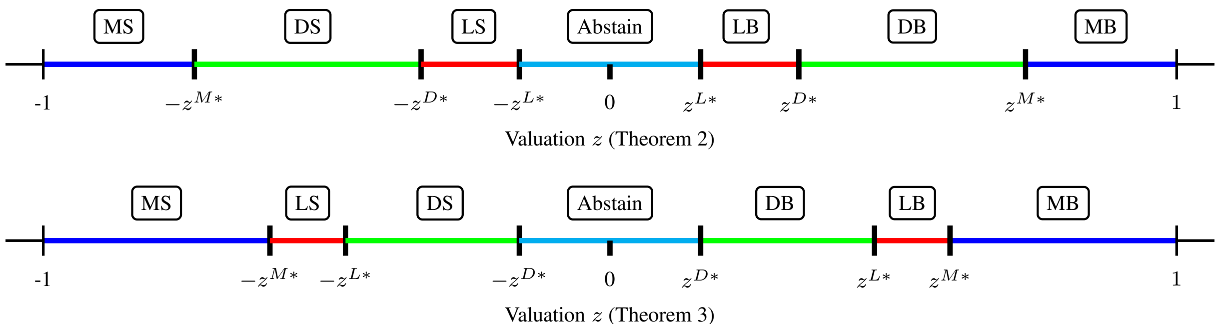

Figure 4: Equilibrium Investor Order Placement. The upper figure illustrates the equilibrium behavior of investors choosing between a lit market, and a dark pool that offers a small price improvement. The lower figure illustrates the case with a large price improvement dark pool. In each case, an investor with valuation $z \in [-1, 1]$ optimally chooses the order type O_t that corresponds to their valuation.

4 Price Improvement, Market Quality and Price Efficiency

4.1 Market Quality and Investor Welfare

A dark pool impacts investor decisions broadly by introducing a new order type so that investors may better segment along their immediacy preferences. Conditional on their valuation z, investors seek a bundle that minimizes execution risk and price impact. In equilibrium, investors with low valuations preference lower price impact over lower execution risk, and invoking Lemma 2, higher valuation investors shift their preferences toward lower execution risk. In this way, the price improvement offered by the dark pool informs the type of investors it attracts: if the price improvement is small, then the relatively low execution risk on offer entices the high immediacy types, whereas a large price improvement attracts the low immediacy types.

I define a benchmark equilibrium in this framework to be the single market equilibrium——hereto referred to as the "lit-only" equilibrium—of Theorem 1, which admits valuation thresholds z_B^M and z_B^L. Using these order placement decisions as a benchmark, I examine the impact of introducing a dark pool on investor order placement strategies, and subsequently, measures of market quality. The following measures present in terms of buy orders, multiplied by 2 to account for sell orders.

An important regulatory concern against the proliferation of dark pools is the impact on lit market liquidity. I capture a measure of this through the quoted spread at the lit market, which is synonymous with the price impact of a market order, given the absence of exogenous transaction costs. A dark pool that offers a small price improvement attracts marginal investors who, in a lit-only setting, would be indifferent to the high fill rate and high price impact market order, and the relatively low fill rate and low price impact alternative offered by limit orders. The monotonicity of investor preferences across their valuations leads marginal market order submitters—who are on average, the least informed market order submitters—to migrate to the dark pool, raising the average informativeness of market orders. Hence, market order price impact increases, widening the quoted spread.

Conversely, a dark pool that offers a large price improvement positions itself as a lower price impact alternative to limit orders such that investors that are more sensitive to price impact than execution risk (i.e., those with the lowest valuations) migrate to the dark pool, increasing the price impact of limit orders. The effect ripples into high-valuation limit order submitters, who migrate to market orders. The result is a decrease in the price impact of market orders, and narrowing of the bid-ask spread.

Proposition 2 (Quoted Spread / Market Order Price Impact) *Compared to the lit-only equilibrium, a market with dark trading according to a small price improvement widens the quoted spread (market order price impact increases). A large price improvement has the reverse effect.*

While liquidity at the lit market may not improve, the dark pool provides opportunities for more investors to find a suitable avenue to trade by introducing another "product" to the immediacy/price impact space of order types. I measure this impact through total order submission, denoted OS_t, defined as twice the likelihood that an investor who arrives at the market in period t submits a buy order of *some type*.

$$OS_t = 2 \times (\mathsf{Pr}(\mathsf{O}_t^* = \mathsf{MB}^*) + \mathsf{Pr}(\mathsf{O}_t^* = \mathsf{DB}_t^*) + \mathsf{Pr}(\mathsf{O}_t^* = \mathsf{LB}_t^*)) \tag{27}$$

Total order submission increases in any equilibrium with positive dark trading, as the introduction of another order type leads the lowest immediacy order type to have a lower price impact than in the lit-only equilibrium. Hence, more investors with low valuations now participate in the market. Order submission by itself, however, provides only an insight into trading opportunities, and not outcomes. I measure the impact on total trading volume, denoted TV_t, by the probability that a trade occurs in a given period t: the probability that an investor submits either a buy or sell market order to the lit market, or an order to the dark pool that gets filled. Total trading volume at period t is then written as:

$$TV_t = 2 \times (\mathsf{Pr}(\mathsf{O}_t^* = \mathsf{MB}_t^*) + l_{t-1}^* \times \mathsf{Pr}(\mathsf{O}_t^* = \mathsf{DB}_t^*)) \tag{28}$$

Proposition 3 summarizes these results below, with numerical illustration found in Figure 6.

Proposition 3 (Volume and Total Order Submission) *Compared to the lit-only equilibrium, the introduction of a dark pool with a small (large) price improvement decreases (increases) lit market volume. For some $\hat{\lambda} \in [\lambda_1, \lambda_2]$, total volume decreases for all $\lambda \in [\lambda_1, \hat{\lambda}]$; a large price improvement always increases total volume. Market participation is weakly greater for any price improvement.*

The impact on lit market volume follows directly from the impact on liquidity: a narrower spread attracts more market orders; the reverse sees volume migrate to the dark pool. As a reduction in spread under the large price improvement regime leads to higher lit market volume, this necessarily implies net volume creation overall, relative to the benchmark. Overall trading volume in the small price improvement case is more complex. While groups of market order submitters migrate to the dark pool (lit volume falls), so do limit order submitters. Thus, the execution risk of dark orders plays a significant role. I find that the net result is a decrease in total volume for sufficiently small price improvement ($\lambda \in [\lambda_1, \hat{\lambda}]$), in which case, the increase in execution risk for investors who migrate from market orders dominates.

The 2010 SEC Concept Release takes as given that dark pools siphon volume away from lit markets, which premises a concern about harm to price efficiency. Proposition 3 predicts, however, that the impact on volume is dependent on the role that dark orders play in the "immediacy-price-impact pecking order"—to borrow from Menkveld, Yueshen, and Zhu (2016). If dark orders are positioned such that they contribute less to (post-trade) price impact than displayed limit orders (i.e., with a large price improvement), then dark pools will lead to net volume creation, contrary to regulatory concerns.

By improving liquidity and creating volume, a dark pool that offers price improvement closer to the midpoint can be beneficial to market quality and thus potentially welfare-improving, but it is not immediate that a dark pool with a small price improvement that siphons order-flow away from the lit market is welfare-reducing. To study the impact on welfare, I define a measure similar to Bessembinder, Hao, and Zheng (2015) that reflects allocative efficiency over private valuations. Because trading on private information is zero-sum between the buyer and the seller, I measure welfare as the expected realizations of private valuations y_t in period t. Welfare is thus the total expected gains from trade to uninformed investors who participate in the market. At the dark pool, gains from trade accrue only from the liquidity taker (the investor), as the liquidity provider has no private valuation. At the lit market, however, I account for the probability that the liquidity providing counterparty is an investor who submitted a limit order at period $t-1$.

The expression for welfare W_t is written as,

$$\begin{aligned} W_t &= 2\cdot \mathsf{Pr}(\mathsf{O}_t^* = \mathsf{DB}_t^*)\cdot l_{t-1}^* \cdot \mathsf{E}[y_t \mid H_t, \mathsf{O}_t^* = \mathsf{DB}_t^*] + 2\cdot \mathsf{Pr}(\mathsf{O}_t^* = \mathsf{MB}_t^*) \\ &\quad \times \left(\mathsf{E}[y_t \mid H_t, \mathsf{O}_t^* = \mathsf{MB}_t^*] - \rho\mathsf{Pr}(\mathsf{O}_{t-1}^* = \mathsf{LS}_{t-1}^*)\cdot \mathsf{E}[y_{t-1} \mid H_{t-1}, \mathsf{O}_t^* = \mathsf{MB}_t^*, \mathsf{O}_t^* = \mathsf{LS}_{t-1}^*]\right) \\ &= 2\cdot\left(\int_{\mathsf{O}_t^*=\mathsf{MB}_t^*} y_t \mathrm{d}y_t + l_{t-1}^* \cdot \int_{\mathsf{O}_t^*=\mathsf{DB}_t^*} y_t \mathrm{d}y_t + \mathsf{Pr}(\mathsf{O}=\mathsf{MS}^*)\cdot \int_{\mathsf{O}_t^*=\mathsf{LB}_t^*} y_t \mathrm{d}y_t\right) \end{aligned} \tag{29}$$

Comparing welfare in (29) the benchmark equilibrium values, I obtain the following result:

Proposition 4 (Investor Welfare) *Compared to the lit-only equilibrium, a dark pool that offers a small (large) price improvement reduces (improves) investor welfare. Moreover, welfare co-moves negatively with the quoted spread and positively with lit market volume.*

The intuition for Proposition 4 follows directly from the impact on lit market volume. As in empirical findings of Hollifield, Miller, Sandås, and Slive (2006) where unfilled limit orders contribute heavily to welfare loss, a large price improvement migrates some investors from limit orders to market orders, where they realize their private values more frequently. The implication is that dark pools that offer significant price improvement may be socially beneficial. When the price improvement is closer to the displayed quote, however, a dark pool reduces welfare by siphoning market order submitters to the dark pool, where they inherit greater execution risk. Moreover, the effect compounds on the welfare of limit order submitters, as the fill rate of investor limit orders subsequently falls. While overall order submission increases, the negative impacts to the lit market dominate, leading to a decline in investor welfare.

4.2 Price Efficiency

Regulatory concerns around the rise of dark trading in the United States and Canada center primarily around the impact on price efficiency, with the worry that segmenting order flow to opaque markets removes the contribution to price discovery that displayed orders provide.

Zhu (2014) predicts that dark pools crossing orders at the mid-quote forces informed traders with short-lived information into lit markets, while drawing uninformed order flow to the dark, leading to improvements in price discovery. Even with price improvement at the dark pool, the effect attenuates, but remains. Here, I contend that an important consideration for the impact of price improvement on price efficiency is the role played by displayed limit orders in competing for investors who seek price improvement over displayed quotes. Recent empirical work by Brogaard, Hendershott, and Riordan (2018) provides evidence that displayed limit orders are the predominant contributors to price discovery, and so it is important to study how the availability of dark trading impacts price discovery through limit orders. Because displayed limit orders also offer price improvement at the cost of execution risk, they present as a natural substitute to dark orders.

I measure (short-run) price efficiency as accuracy with which an innovation is impounded into the public value through an order's price impact. More formally, the measure PD_t is constructed as the per-period absolute difference between the innovation's value, and the investor's (ex-ante) expected price impact.

$$PD_t = \mathsf{E}[|(\delta_t - (1-\mu) \times \mathsf{E}[\delta_t \mid \mathsf{uninformed}, \mathsf{O}_t^*] - \mu \times \mathsf{E}[\delta_t \mid \mathsf{informed}, \mathsf{O}_t^*])|] \tag{30}$$

Note that uninformed investors have mean-zero contribution to price efficiency, as their private valuation is uncorrelated with δ_t, yielding $\mathsf{E}[\delta_t \mid \mathsf{uninformed}, \mathsf{O}_t] = 0$. Moreover, for any value δ_t, the corresponding price impact must be lower, by the equilibrium condition that informed investors make non-negative profit when trading on information, and hence the absolute value bars can be removed. I simplify Equation (30) into three terms that correspond to the errors of each order type.

$$PD_t = 1 - 2\mu \left(\int_{\mathsf{O}_t^* = \mathsf{MB}_t^*} \mathsf{E}[\delta_t \mid \mathsf{O}_t^*] \mathrm{d}\delta_t + \int_{\mathsf{O}_t^* = \mathsf{LB}_t^*} \mathsf{E}[\delta_t \mid \mathsf{O}_t^*] \mathrm{d}\delta_t + l_{t-1}^* \cdot \int_{\mathsf{O}_t^* = \mathsf{DB}_t^*} \mathsf{E}[\delta_t \mid \mathsf{O}_t^*] \mathrm{d}\delta_t \right) \tag{31}$$

As a departure from Zhu (2014), I allow innovations to the public value to vary in impact by drawing δ_t from a symmetric distribution. This induces informed investors to use an order type that reflects the degree of urgency with respect to the value of their information: the greater the impact, the more profitable the information, and thus, the lower tolerance for execution risk when selecting an optimal order type.

Proposition 5 (Price Efficiency) *Compared to the lit-only equilibrium, an investor's expected price impact conditional on the fill of their order improves; however, price efficiency (unconditional expected price impact) worsens in any equilibrium with positive dark trading.*

In comparison to the lit-only equilibrium, a dark pool allow investors to better partition across a wider array of order types, which moves conditional price impact closer to the innovation's expected value (see Figure 7). However, orders routed to the dark contribute to price discovery only when filled; if the order is cancelled, the order is never observed by other market participants. This is contrary to displayed limit orders, which contribute to price discovery simply by the price improvement they provide when posting to the lit market. The result is that their unconditional price impact is skewed by the this execution risk, regardless of the position of dark orders in the immediacy hierarchy. This latter effect dominates the improvement in conditional price impact, leading to worse price efficiency when investors use the dark pool.

As a caveat, I caution that the implications for price efficiency should only be taken to reflect the short-run. Because innovations to the public value are never fully revealed in my model (except on market close), $Var(V_t \mid H_t)$ increases in t. For this reason, my results necessarily rely on the risk-neutral assumption imposed on market participants; results in a model with risk-averse liquidity providers may differ.

4.3 Price Improvement Selection by Dark Pools

In equilibrium, a dark pool is active for a wide a range of price improvement levels. In this section I explore how a dark pool might select an "optimal" price improvement, by assuming that a dark pool seeks to maximize its profit through trading volume. This reflects the reality of exchanges to charge per-trade access fees to participants.[14] Through this lens, I examine two dark pool environments: i) the dark pool is an independent venue (e.g., Crossfinder, UBS), and ii) the dark pool operates as a subsidiary of a parent lit market (e.g., TSX Alpha Intraspread).

Proposition 6 (Price Improvement Selection) *Compared to the lit-only equilibrium, introducing a dark pool as an independent venue decreases lit market volume, widens the visible quoted spread, and worsens investor welfare; a dark pool that is operated by the lit market has the reverse effect. Moreover, an independent dark pool maximizes volume at* $\lambda = \lambda_1$.

An independent dark pool seeks only to maximize their own volume. Hence, the dark pool maximizes the expression: $2(l^* \cdot \mathsf{Pr}(\mathsf{DB}))$. Figure 6 provides a numerical example, illustrating that the profit-maximizing value λ falls into the small price improvement range, $\lambda \in (0, \lambda^*)$, as total dark volume (blue line) is higher than in any equilibrium with a large price improvement. Thus, by Propositions 2-4, introducing an independent dark pool alongside a lit market reduces market quality and investor welfare. Moreover, the price improvement level λ_1 that maximizes dark volume leads the dark pool to surpass the lit market in trading volume. Comparatively, a lit market that operates a dark pool will seek to maximize total volume across both markets, $2(l^* \cdot \mathsf{Pr}(\mathsf{DB}) + \mathsf{Pr}(\mathsf{MB}))$. Figure 6 illustrates that total volume (purple line) maximizes with a large price improvement level, which corresponds to improved market quality and investor welfare,

[14] See e.g., Colliard and Foucault (2012), Malinova and Park (2015), and others.

when compared to a lit-only environment. This result has important implications for current policy on limiting dark volume. As of January of 2018, the European Union imposes a percent-of-total-volume cap on dark pools. The cap mandates that any single dark pool may execute no more than 4% of total trades in any security for a 12-month period (see Petrescu and Wedow (2017)). Thus, the cap has the potential to reduce, or even eliminate the negative impact of dark pools that maximize dark volume at the expense of lit markets.

Proposition 6 highlights that intermediated dark pools may, in their own interest, provide an important step towards optimal allocative efficiency in equity markets; this, however, depends on whether the interests of dark pools and lit markets are aligned. If dark pools internalize their potential negative impact on lit market volume (e.g., the dark pool is operated by the lit market), then dark orders find a socially beneficial role in the market. If maximizing market share is the sole objective—to the detriment of lit markets—then a case can be made for regulators to introduce a minimum price improvement rule that relegates dark pools to the bottom of the immediacy hierarchy. As a caveat, my results examine competition between a single intermediated dark pool and a single lit market, abstracting from inter-dark pool competition. Thus, my results may over-estimate the impact of any one dark pool on the overall market.

5 Empirical Predictions

My model generates several testable predictions on the impact of intermediated dark pools. Moreover, many predictions can be tested from two angles: using the relative price impacts of limit order submissions and dark trades, or the relative fill rates of limit orders and dark orders. Firstly, as the dark pool accepts orders from all investors, the liquidity provider is subject to adverse selection, and thus requires a minimum spread to break even. Hence, my model predicts that any rule that forces price improvement to the midpoint ($\lambda = 1/2$) will drive liquidity providers from the dark pool, eliminating any intermediation.

Empirical Prediction 1 (Intermediation in Dark Pools) *Any dark pool that mandates midpoint crossing* ($\lambda = 1/2$) *in the presence of informed traders* ($\mu > 0$) *eliminates intermediation at the dark pool* ($l^* = 0$).

Prediction 1 has important implications for the global regulatory debate on minimum price improvement rules (MPIR)—also known as "trade-at rules"—as rules that are too restrictive may eliminate intermediated dark trading entirely. On October 15th, 2012, Canadian regulators implemented a trade-at rule that requires most dark orders to provide a meaningful price improvement of one trading increment; the rule is a midpoint-crossing rule for spreads of one or two ticks.[15] The Australian Securities and Investments Commission (2012) followed with a similar rule on May 26th, 2013. Moreover, in the U.S., the Securities and Exchange Commission (2014) has included a similar trade-at rule component in their recent pilot program on the role of tick size in equity markets.[16] Prediction 1 has support from empirical studies on the Canadian trade-at rule. Comerton-Forde, Malinova, and Park (2018) and Foley and Putniņš (2016) find that two-sided liquidity provision sharply declined following the introduction of the MPIR in Canada.[17] Particularly, Comerton-Forde, Malinova, and Park (2018) find that following the MPIR, market maker liquidity at the two largest Canadian dark pools drops from 88% and 18%, to 38% and 1%, respectively.

Given that fill rates of displayed limit orders (also synonymous with the trading rate) relative to (marketable) orders sent to the dark pool informs the qualitative impact of dark trading on market quality, welfare, and price efficiency, the impact of a midpoint trade-at rule has the effect of eliminating intermediated dark trades, and thus their impact on the market. This yields the following prediction:

Empirical Prediction 2 (Midpoint Trade-at Rule) *If the displayed limit order fill rate is less than the fill rate of dark market orders, a midpoint a trade-at rule ($\lambda = 1/2$) will:*

(1) *narrow quoted spreads, lower price impact, and increase lit and total market volume.*

(2) *improve price efficiency.*

If the limit order fill rate exceeds the dark market order fill rate, then prediction (1) is reversed.

Comerton-Forde, Malinova, and Park (2018) document that on the dark pool with the largest proportion of market-maker participation, the Canadian MPIR correlated with a reduction in dark execution probability of 76.4%. Using the same event, Foley and Putniņš (2016) find that the implementation of the MPIR on two-sided dark markets worsened quoted, effective and realized spreads, an effect that my model suggests is a result of dark market orders being lower than displayed limit orders in the urgency hierarchy. Moreover, when the estimation of execution probabilities for dark market orders and displayed limit orders is not possible, Corollary 1 provides that Empirical Prediction 2 can be re-written in terms of dark order and limit order price impacts.

[15] See Investment Industry Regulatory Organization of Canada (2011).

[16] The SEC pilot program permits retail orders to receive less than one tick price improvement, but other orders must trade at a minimum full tick price improvement or the midpoint—the latter is often the case for highly liquid securities.

[17] Rosenblatt Securities Inc. (2013) notes that the immediate impact of the Canadian trade-at rule was a 50% drop in dark volume.

Finally, the role of midpoint trade-at rules as inhibitors to dark pool intermediation may depend on the state of market organization at their implementation; that is, whether dark pools are largely independently-operated, or function as subsidiaries of lit markets.

Empirical Prediction 3 (Venue Competition) *If an intermediated dark pool operates independently of a lit market, a midpoint price improvement rule will improve market quality; if it is operated by a lit market, then such a rule will worsen market quality.*

6 Conclusion

In dark pools, investors can earn price improvement on the prevailing spread, but at increased execution risk. Hence, investors with lower immediacy demands may segment away from lit markets. Because displayed limit orders also offer price improvement and execution risk, I argue that they provide a natural substitute to dark orders, and thus play a key role in the impact of dark trading on equity markets.

In this paper, dark pools source liquidity from competitive market makers and offer a set price improvement to investors. Within this framework, I show that investors' trading strategies form an immediacy hierarchy: investors with valuations further from the public value select order types that limit execution risk. The key insight is that the relative ranking of dark orders and limit orders in this hierarchy dictates the impact of dark pools on market quality and welfare. Moreover, the ranking is informed by the level of price improvement at the dark pool, as well as the impacts to market quality and welfare.

In absence of regulation, an independently-operated dark pool offers price improvement that maximizes their own volume to the detriment of the lit market, consequently reducing welfare. Thus, a case can be made for minimum price improvement rules or per-venue dark volume caps as useful tools to improve investor welfare. For example, a price improvement rule that forces all orders to cross at midpoint—similar to those in Canada and Australia—will eliminate all intermediation from dark pools, which impacts markets favourably when dark pools operate as independent venues. Per-venue volume caps may also improve market quality, by aligning the profit-maximization interest of dark pools with total volume. I caution that while dark trading may improve liquidity and welfare, these benefits come at a cost of worse price efficiency, as any price improvement rule that leads to positive dark volume attracts order flow away from lit markets—both active and passive—that would have otherwise contributed to price efficiency.

References

Anderson, Lisa, Baiju Devani, and Yifan Zhang, 2015, Impact of the dark rule amendments, Discussion paper, IIROC.

Australian Securities and Investments Commission, 2012, 12-290mr: Asic makes key announcements on market structure, dark liquidity and high frequency trading, Discussion paper, ASIC.

Bessembinder, Hendrik, Jia Hao, and Kuncheng Zheng, 2015, Market making contracts, firm value, and the ipo decision, *Journal of Finance* 70, 1997–2028.

Bloomfield, Robert J., Maureen O'Hara, and Gideon Saar, 2015, Hidden liquidity: Some new light on dark trading, *Journal of Finance* 70, 2227–2274.

Boulatov, Alex, and Thomas J. George, 2013, Hidden and displayed liquidity in securities markets with informed liquidity providers, *Review of Financial Studies* 26, 1027–1063.

Brogaard, Jonathan, Terrence Hendershott, and Ryan Riordan, 2018, Price discovery without trading: Evidence from limit orders, *Journal of Finance (accepted)*.

Brolley, Michael, and Katya Malinova, 2013, Informed trading and maker-taker fees in a low-latency limit order market, *SSRN eLibrary*.

Budish, Eric, Peter Cramton, and John Shim, 2015, The high-frequency trading arms race: Frequent batch auctions as a market design response, *Quarterly Journal of Economics* 130, 1547–1621.

Buti, Sabrina, and Barbara Rindi, 2013, Undisclosed orders and optimal submission strategies in a limit order market, *Journal of Financial Economics* 109, 797–812.

———, and Ingrid Werner, 2011, Diving into dark pools, *SSRN eLibrary*.

———, 2017, Dark pool trading strategies, market quality and welfare, *Journal of Financial Economics* 124, 244–265.

CFA Institute, 2012, Dark pools, internalization, and equity market quality, Discussion paper, CFA Institute http://www.cfapubs.org/doi/pdf/10.2469/ccb.v2012.n5.1.

Colliard, Jean-Edouard, and Thierry Foucault, 2012, Trading fees and efficiency in limit order markets, *Review of Financial Studies* 25(11), 3389–3421.

Comerton-Forde, Carole, Katya Malinova, and Andreas Park, 2018, Regulating dark trading: Order flow segmentation and market quality, *Journal of Financial Economics (forthcoming)*.

Comerton-Forde, Carole, and Tālis Putniņš, 2015, Dark trading and price discovery, *Journal of Financial Economics* 118, 70–92.

Degryse, Hans, Frank de Jong, and Vincent van Kervel, 2015, The impact of dark trading and visible fragmentation on market quality, *Review of Finance* 19, 1587–1622.

Degryse, Hans, Nikolaos Karagiannis, Geoffrey Tombeur, and Gunther Wuyts, 2018, Two shades of opacity: Hidden orders versus dark trading, *SSRN eLibrary*.

Degryse, Hans, Mark van Achter, and Gunther Wuyts, 2009, Dynamic order submission strategies with competition between a dealer market and a crossing network, *Journal of Financial Economics* 91, 319 – 338.

Foley, Sean, and Tālis Putniņš, 2016, Should we be afraid of the dark? dark trading and market quality, *Journal of Financial Economics* 122, 456–481.

Foucault, Thierry, 1999, Order flow composition and trading costs in a dynamic limit order market, *Journal of Financial Markets* 2, 99–134.

Glosten, Lawrence, 1994, Is the electronic open limit order book inevitable?, *The Journal of Finance* 49, 1127–1161.

——— , and Paul Milgrom, 1985, Bid, ask and transaction prices in a specialist market with heterogenously informed traders, *Journal of Financial Economics* 14, 71–100.

Hendershott, Terrence, and Haim Mendelson, 2000, Crossing networks and dealer markets: Competition and performance, *Journal of Finance* 55, 2071–2115.

Hollifield, Burton, Robert A. Miller, and Patrik Sandås, 2004, Empirical analysis of limit order markets, *Review of Economic Studies* 71, 2096–2137.

———, and Joshua Slive, 2006, Estimating the gains from trade in limit order markets, *Journal of Finance* 611, 2753–2804.

Investment Industry Regulatory Organization of Canada, 2011, Provisions respecting dark liquidity, iiroc notice 11-0225, Discussion paper, IIROC http://www.iiroc.ca/Documents/2012/77c0af22-004e-417d-9217-a160b3fcb5c5_en.pdf.

Iyer, Krishnamurthy, Ramesh Johari, and Ciamac C. Moallemi, 2018, Welfare analysis of dark pools, *SSRN eLibrary*.

Kaniel, Ron, and Hong Liu, 2006, So what orders so informed traders use?, *Journal of Business* 79, 1867–1913.

Malinova, Katya, and Andreas Park, 2015, Subsidizing liquidity: The impact of make/take fees on market quality, *Journal of Finance* 70.

Menkveld, Albert J., Bart Zhou Yueshen, and Haoxiang Zhu, 2016, Shades of darkness: A pecking order of trading venues, *Journal of Financial Economics (forthcoming)*.

Milgrom, Paul, and Nancy Stokey, 1982, Information, trade and common knowledge, *Journal of Economic Theory* 26, 17–27.

Mittal, Hitesh, 2008, Are you playing in a toxic dark pool? a guide to preventing information leakage, *Journal of Trading* 3, 20–33.

Moinas, Sophie, 2011, Hidden limit orders and liquidity in order driven markets, Discussion paper, TSE.

Nimalendran, Mahendrarajeh, and Sugata Ray, 2014, Informational linkages between dark and lit trading venues, *Journal of Financial Markets* 17, 230–261.

Parlour, Christine, 1998, Price dynamics in limit order markets, *Review of Financial Studies* 11, 789–816.

Patterson, Scott, 05-06-2013, ‘dark pools’ face scrutiny, *The Wall Street Journal*.

Petrescu, Monica, and Michael Wedow, 2017, Dark pools in european equity markets: Emergence, competition and implications, Discussion paper, European Central Bank.

Ready, Mark, 2014, Determinants of volume in dark pool crossing networks, *SSRN eLibrary*.

Rosenblatt Securities Inc., 2013, Let there be light: February 26, *Trading Talk*.

———, 2016, Let there be light: June 23, *Trading Talk*.

Rosu, Ioanid, 2009, A dynamic model of the limit order book, *Review of Financial Studies* 22, 4601–4641.

Securities and Exchange Commission, 2010, Concept release on market structure, Release No. 34-61358, File No. S7-02-10, Discussion paper, http://www.sec.gov/rules/concept/2010/34-61358.pdf.

———, 2014, SEC announces order for tick size pilot plan, Discussion paper, http://www.sec.gov/News/PressRelease/Detail/PressRelease/1370542172819.

Weaver, Daniel, 2014, The trade-at rule, internalization, and market quality, *SSRN eLibrary*.

Ye, Linlin, 2017, Understanding the impacts of dark pools on price discovery, Discussion paper, .

Ye, Mao, 2011, A glimpse into the dark: Price formation, transaction costs, and market share in the crossing network, Discussion paper, University of Illinois.

Zhu, Haoxiang, 2014, Do dark pools harm price discovery?, *Review of Financial Studies* 27, 747–789.

A Appendix

A.1 List of Variables and Parameters

Variable	Description
V_t	asset fundamental value at period t
v_t	public value at period t
μ	total mass of informed traders
δ_t	innovation to the fundamental asset value at period t
y_t	private value of a liquidity investor that arrives at period t
z_t	valuation of an investor who arrives at period t
ρ	probability of market close at the end of current period
λ	required price improvement (percentage of the bid-ask spread) for (market) orders sent to the dark pool.
H_t	public information at period t
ask_t	ask price at the lit market at period t
bid_t	bid price at the lit market at period t
inv	denotes an investor t
LP	denotes a liquidity provider
z^M	valuation below which an investor will not submit a market buy order
z^L	valuation below which an investor will not submit a limit buy order
z^D	valuation below which an investor will not submit a dark buy order
B	subscript denoting an "lit-only" equilibrium value (e.g., z_B^M)
l_t	dark pool liquidity provision intensity by liquidity provider at period t
O_t	period t investor's action
MB_t	period t investor submits a market buy order
MS_t	period t investor submits a market sell order
LB_t	period t investor submits a limit buy order at limit price bid_{t+1}
LS_t	period t investor submits a limit sell order at limit price ask_{t+1}
DB_t	period t investor submits a dark buy order
DS_t	period t investor submits a dark sell order
NT_t	period t investor abstains from trade

A.2 Proofs: Lemmas

Proof (Lemma 1). First, I show that for any buy order, an investor's payoff function can be broadly characterized in terms of an investor's valuation, the order's execution probability and price impact, as in (16). Sell order types hold by symmetry. For a market buy order, we take the payoff to a market buy order from (1), and substitute the ask_t^* from (7).

$$\pi_{\mathsf{MB}} = v_t + z_t - \mathsf{ask}_t = z_t - \mathsf{E}[\delta_t \mid \mathsf{O}_t = \mathsf{MB}_t] \tag{32}$$

where $\mathsf{E}[\delta_t \mid \mathsf{O}_t = \mathsf{MB}_t]$ describes a market buy order's price impact. Now, consider the investor's payoff to a limit buy order, given by (2). By substituting bid_{t+1}^* from (8), I simplify the payoff function to:

$$\pi_{\mathsf{LB}} = \rho \mathsf{Pr}(\mathsf{O} = \mathsf{MS}^*) \times (\mathsf{E}[V_{t+1} \mid \mathsf{O}_t = \mathsf{LB}_t, \mathsf{O}_{t+1} = \mathsf{MS}_{t+1}] - \mathsf{bid}_{t+1}^*) \tag{33}$$

$$= \rho \mathsf{Pr}(\mathsf{O} = \mathsf{MS}^*) \times (z_t - \mathsf{E}[\delta_t \mid \mathsf{O}_t = \mathsf{LB}_t]) \tag{34}$$

similar to a market order, $\mathsf{E}[\delta_t \mid \mathsf{O}_t = \mathsf{LB}_t]$ describes a limit buy order's price impact. Lastly, an investor's contribution to the price impact of a dark order follows from the fact that the liquidity provider's zero-profit condition implies $\mathsf{ask}^{\text{Dark}*} = \mathsf{E}[\delta_t \mid \mathsf{O}_t = \mathsf{DB}_t]$. Using this, I rewrite the investor's dark order payoff (3) as,

$$\pi_{\mathsf{DB}} = l^* \times (v_t + z_t - v_t - (1 - 2\lambda)\mathsf{E}[\delta_t \mid \mathsf{O}_t = \mathsf{MB}_t]) = l^* \times (z_t - \mathsf{E}[\delta_t \mid \mathsf{O}_t = \mathsf{DB}_t]) \tag{35}$$

which reflects the formulation outlined in (16).

Now, I show that, in a stationary, symmetric equilibrium, it must be the case that any investor who submits a buy order has $z_t \geq 0$, and symmetrically for sell orders. Time subscripts are dropped, as I focus on stationary equilibria. Let γ_I denote the fill rate of order type, I, and let p_I denote order type I's price impact. Further, let investor t have a valuation equal to $z_t \geq 0$. For any order type I, there is a buy and a sell option, IB, and IS, respectively. Then, for any investor z_t, a buy order of type I is preferred to a sell order of type I if and only if:

$$\gamma_{\mathsf{IB}} \times (z_t - p_{\mathsf{IB}}) \geq \gamma_{\mathsf{IS}} \times (z_t - p_{\mathsf{IS}}) \tag{36}$$

In any symmetric equilibrium, $\gamma_{\mathsf{IB}} = \gamma_{\mathsf{IS}}$, and $p_{\mathsf{IB}} = -p_{\mathsf{IS}}$. Then, (36) becomes $z_t \geq 0$, implying that no investor with $z_t \geq 0$ would prefer IS to IB.

I can now show that for any buy order type IB to be used in equilibrium, the price impact p_{IB} must be positive. To see this, suppose instead that $p_{\mathsf{IB}} < 0$. It must be then, by symmetry, that $p_{\mathsf{IS}} > 0$. Now, because

p_{IB} describes the average informativeness of investors who submit orders of type IB in equilibrium, it must be the case that some investor with $z_t < 0$ submits orders of type IB. But from the previous argument, any investor with $z_t < 0$ must prefer IS to IB. A contradiction. Thus, in any equilibrium, buy orders that are used by some investors must have a positive price impact, $p_{\mathsf{IB}} > 0$, and symmetrically for sell orders.

Lastly, I show that investors who use buy orders of type I do not prefer any other sell order type J. Consider two order types, I and J. Again, symmetry allows us to consider a buy order of type I, and a sell order of type J, with the reverse following analogously. That investors with $z_t \geq 0$ will not use any sell order type, in equilibrium, follows from the argument above. Suppose that an investor $z_t \geq 0$ prefers a sell order of type JS to a buy order, IS. By the argument above, this investor *must* prefer order type JB to JS. Note, finally, that we have only shown that investors with $z_t \geq 0$ do not submit sell orders in equilibrium. Because $p_{\mathsf{IB}} > 0$, any investor with $z_t \in (0, p_{\mathsf{IB}})$ will prefer to abstain from trading, or prefer another buy order type J, with $p_{\mathsf{JB}} < p_{\mathsf{IB}}$.

The argument for investors not using buy orders if $z_j \leq 0$ follows by symmetry. ■

Proof (Lemma 2). Let γ_I denote the fill rate of order type, I, and let p_I denote I's price impact. In this proof, the buy and sell notation is dropped, as I focus on the decisions between buy order types. The decisions for sell order types proceeds analogously. I will show that in any symmetric, stationary equilibrium where all order types are used, investors use a threshold strategy such that an investor with valuation $z_t \geq z^J > z^K$ prefers order type J to order type K. Moreover, if $\gamma_J \geq \gamma_K$, then $z^J \geq z^K$ for all p_J, p_K.

Suppose there is an order type J such that, in equilibrium, $\gamma_J \geq \gamma_K$ and $p_J < p_K$. Suppose order type K be used by some investor, t, in equilibrium (investor t earns non-negative profits). Then, the profit of investor t from using order type K is,

$$\pi_K^t = \gamma_K \times (z_t - p_K) < \gamma_J \times (z_t - p_J) = \pi_J^t \tag{37}$$

But this contradicts the supposition that investor t prefers order type K. Thus, for any order used in equilibrium, no other order can have both a higher fill rate, and a lower price impact. Hence, for I, J, and K to be used in equilibrium, it must be that $\gamma_I \geq \gamma_J \geq \gamma_K \geq 0$ implies $p_I \geq p_J \geq p_K \geq 0$. By corollary,

$$\gamma_I p_I \geq \gamma_J p_J \geq \gamma_K p_K \geq 0 \tag{38}$$

Recall that the profit function of order type I for investor t, $\pi_I^t = \gamma_I \times (z_t - p_I)$, is linear in z_t; moreover, the order with the highest fill rate also has the lowest intercept, $-\gamma_I p_I$. The linearity of π_I in z_t implies that since $\gamma_I \geq \gamma_J \geq \gamma_K \geq 0$, if π_I crosses π_J at some z_T, it remains above π_J for all $z_t > z_T$. Similarly for π_K. Moreover, because $\gamma p_I \geq \gamma_J p_J \geq \gamma_K p_K$, for order type I to be used in equilibrium, π_I must cross both π_J and π_K. Thus, $\max\{z_J, z_K\} \leq z_I \leq 1$.

Then, the relation in (38) implies that for I to be used in equilibrium, it must cross π_J and π_K at some $z_I < 1$. Likewise, since $\gamma_J \geq \gamma_K \geq 0$, for J to be used in equilibrium, π_J must cross π_K, before π_I crosses π_J. Thus, $z_J \leq z_I$. Lastly, π_K crosses the no-trade threshold, $\pi_{NT} = 0$ before π_J and π_I, but at a positive $z_K = \gamma_K p_K$. Hence, $z_K > 0$, and $z_K \leq z_J \leq z_I$. ■

A.3 Proofs: Theorems

The proofs of Theorems 1-3 proceed similarly. I select a reference threshold, for which I show that all other thresholds exist and are unique for all values of the reference threshold, making use of function continuity, the intermediate value theorem, Lemma (2), and the implicit function theorem. In the final step, I show that there exists a unique reference threshold. I drop time subscripts in all proofs, by assumption of stationarity.

Proof (Theorem 1). In the lit-only equilibrium, there are only two order types: limit orders, and market orders. I denote the lit-only equilibrium thresholds $z^{L*} = z^L_B$ and $z^{M*} = z^M_B$. The subscript 'B' is used throughout the proof section to denote lit-only (benchmark) equilibrium values. To prove existence and uniqueness of a lit-only equilibrium, recall the two equilibrium payoff conditions from (17) and (18):

$$z^M - \frac{\mathsf{E}[\delta \mid z \geq z^M] - \rho \cdot \mathsf{Pr}(z \leq -z^M) \times \mathsf{E}[\delta \mid z^L \leq z < z^M]}{1 - \rho \cdot \mathsf{Pr}(z \leq -z^M)} = 0 \tag{39}$$

$$z^L - \frac{\mu z^M}{2 - \mu} = 0 \tag{40}$$

Denote the left-hand sides of (39) and (40) by $\Delta^M_B(z^L, z^M)$ and $\Delta^L_B(z^L, z^M)$, respectively. By inspection, $z^L_B = \frac{\mu z^M}{2-\mu} \in (0, z^M)$ solves (40) for all $z^M \in [0, 1]$. I now show that $\Delta^M_B(z^L_B(z^M), z^M)$ must crosses zero from below at least once. First substitute z^L_B into (39). Then inputting bounds 0 and 1:

$$\Delta^M_B(z^L_B(0), 0) = 0 - \mathsf{E}[\delta \mid z \geq z^M] - \rho \mathsf{Pr}(z \leq -z^M) \times (0 - 0) < 0 \tag{41}$$

$$\Delta^M_B(z^L_B(1), 1) = 1 - \mathsf{E}[\delta \mid z \geq z^M] - 0 \times \left(1 - \frac{\mu}{2 - \mu}\right) > 0 \tag{42}$$

Because Δ^M_B is continuous in z^M, the intermediate value theorem implies the existence of a $z^M_B \in [0, 1]$ that crosses zero from below for all z^L_B. This holds for z^L_B by the implicit function theorem, as $\Delta^L_B(\cdot, \cdot)$ is continuous for all z_B^M. Then, taking the first derivative of $\Delta_B^M(\cdot,\cdot)$ with respect to z_B^M,

$$\frac{\partial \Delta_B^M}{\partial z_B^M} = 1 - \frac{\mu}{2} + \frac{\rho}{2} \cdot (z_B^M - \mathsf{E}[\delta \mid z_B^L \leq z < z_B^M]) - \rho \mathsf{Pr}(z \leq -z_B^M) \times \left(1 - \frac{\mu}{2-\mu}\right) \tag{43}$$

$$= \frac{2(1-\mu) + \mu^2 + 4\rho(1-\mu)z_B^M + 2(1-\mu)\cdot(1-\rho)}{2(2-\mu)} > 0 \tag{44}$$

Therefore, a unique equilibrium exists in threshold strategies. Lastly, because z_B^M is the solution to a quadratic function with only one positive root, I can solve (39) for z_B^M explicitly:

$$z_B^M = \frac{\sqrt{(2-\mu)^4 - 4(1-\mu)\rho(3\mu^2 - (8-\rho)\mu + 4 - \rho)} - 2(2-\rho)(1-\mu) - \mu^2}{4\rho(1-\mu)} \tag{45}$$

Optimality of Investor and Liquidity Provider Strategies. In equilibrium, investors submit market orders, limit orders at competitive prices, or abstain from trading. An investor's deviation from one equilibrium action to another equilibrium action will not affect equilibrium bid and ask prices or probabilities of the future order submissions. By Lemma 2, investors cannot profitably deviate from the prescribed equilibrium actions, based on threshold decision rules, when their choices are restricted to one of the following actions: submit a market order (to buy or to sell), submit a limit order at the prescribed competitive equilibrium bid or ask price, or abstain from trading. If an investor submits a limit order at a price off-the-equilibrium path, the liquidity provider reacts to mitigate any incentive for subsequent investors to deviate from the equilibrium strategy. I detail the liquidity provider's response to off-equilibrium actions in section A.5. ■

Proof (Theorem 2). I first show that for any equilibrium satisfying Definitions (1)-(2), $\lambda \in (0, \lambda^*)$ if and only if the equilibrium threshold values satisfy $0 \leq z^{L*} \leq z_B^L \leq z^{D*} \leq z_B^M \leq z^{M*} \leq 1$. Then, similar to the proof of Theorem 1, I proceed by proving the existence and uniqueness of z^{L*}, z^{D*} and l^* for all z^M, and then show that a unique z^{M*} exists for some $\lambda \in (0, \lambda^*)$. Throughout, I recall conditions (19)-(22):

$$\Delta_{\lambda<\lambda^*}^l \equiv z^M - \mathsf{E}[\delta \mid z \geq z^M] - l \cdot \left(z^M - (1-2\lambda) \times \mathsf{E}[\delta \mid z \geq z^M]\right) \tag{46}$$

$$\Delta_{\lambda<\lambda^*}^M \equiv l \cdot \left(z^D - (1-2\lambda)\mathsf{E}[\delta \mid z \geq z^M]\right) - \rho \mathsf{Pr}(z \leq -z^M) \times \left(z^D - \mathsf{E}[\delta \mid z^L \leq z < z^D]\right) \tag{47}$$

$$\Delta_{\lambda<\lambda^*}^L \equiv z^L - \mathsf{E}[\delta \mid z^L \leq z < z^D] \tag{48}$$

$$\Delta_{\lambda<\lambda^*}^D \equiv (1-2\lambda)\mathsf{E}[\delta \mid z \geq z^M] - \mathsf{E}[\delta \mid z^D \leq z < z^M] \tag{49}$$

Moreover, in what follows, I make the notational substitution: $\mathsf{E}[\delta \mid z_I \leq \delta < z_J] \equiv f(z_I, z_J) = \frac{\mu(z_I + z_J)}{2}$. It is immediate that $f(z_I, z_J)$ is increasing in both z_I and z_J.

Marginal Valuation Threshold Ranking

First, I show that any equilibrium values z^{D*}, z^{L*}, and z^{M*} that solve (46)-(49) must satisfy $0 \leq z^{L*} \leq z_B^L \leq z^{D*} \leq z_B^M \leq z^{M*} \leq 1$. Rewriting conditions in (19)-(22) in terms of $f(z_I, z_J)$ yields:

$$z^{M*} - f(z^{M*}, 1) - l \cdot (z^{M*} - f(z^{D*}, z^{M*})) = 0 \tag{50}$$

$$l \cdot (z^{D*} - f(z^{D*}, z^{M*})) - \rho \mathsf{Pr}(z \leq -z^{M*}) \times (z^{D*} - f(z^{L*}, z^{D*})) = 0 \tag{51}$$

$$z^{L*} - f(z^{L*}, z^{D*}) = 0 \tag{52}$$

Now, let $z^{M*} < z_B^M$ and $z^{D*} < z_B^L$. Then it must be the case that $l^* > \rho \mathsf{Pr}(z \leq -z^M) > \rho \mathsf{Pr}(z \leq -z_B^M)$, and $f(z^{D*}, z^{M*}) < f(z_B^L, z_B^M)$. Moreover,

$$l^* \times (z^{M*} - f(z^{D*}, z^{M*})) > \rho \mathsf{Pr}(z \leq -z_B^M) \times (z^{M*} - f(z_B^L, z_B^M)) \tag{53}$$

This implies that any z^{M*} that solves (50) satisfies $z^{M*} > z_B^M$, a contradiction. Moreover, $f(z^{L*}, z^{D*}) \leq f(z_B^L, z_B^M)$ implies that to satisfy $\pi_M(z^{M*}) \geq \pi_L(z^{M*})$, $z^{M*} \geq z_B^M$. Thus, $z^{M*} \geq z_B^M$.

Now, let $z^{D*} > z_B^M$. If we look for an equilibrium where $z^{M*} = z_B^M$, we can rearrange (50)-(52) to form the following equation:

$$z_B^M - f(z_B^M, 1) - \rho \mathsf{Pr}(z \leq -z_B^M) \times (z^{D*} - f(z^{L*}, z^{D*})) \frac{(z_B^M - f(z^{D*}, z_B^M))}{(z^{D*} - f(z^{D*}, z_B^M))} = 0 \tag{54}$$

By inspection, $z^{M*} = z^{D*} = z_B^M$ and $z^{L*} = z_B^L$ form an equilibrium (the lit market equilibrium). Using this fact, I examine what properties an equilibrium must have if $z^{M*} = z_B^M + \epsilon$, where $\epsilon > 0$. Inputting this into equation (50), we have that:

$$z_B^M - f(z_B^M, 1) - l_B \cdot (z_B^M - f(z^{D*}, z_B^M)) + (1 - l_B) \left(1 - \frac{\mu}{2}\right) \epsilon > 0 \tag{55}$$

Thus, in equilibrium, either i) $z^{D*} < z_B^M$, or ii) l^* must be larger than in the case where $z^{M*} = z_B^M$. If the answer is i), we are done. Suppose $z^{D*} \geq z_B^M$, and instead, that $l^* = l_B + \eta$, where $\eta > 0$, such that (55) equals zero. Then, input z^{M*} and l^* into equation (51) to yield equation h_1:

$$h_1 = -(l_B + \eta) \left(z^{D*} - f(z^{D*}, z_B^M) - \frac{\epsilon\mu}{2}\right) - \left(\rho \mathsf{Pr}(z \leq -z_B^M) - \frac{\epsilon}{2}\right) (z^{D*} - f(z^{L*}, z^{D*})) \tag{56}$$

Equation (56) contains the expression (51) when the system is solved for $z^{M*} = z^{D*} = z_B^M$ and $z^{L*} = z_B^L$, and thus is equal to zero. Expression (56) simplifies to:

$$h_1 = -l_B \cdot \frac{\mu\epsilon}{2} + \eta \cdot \left(z^{D*} - f(z^{D*}, z_B^M) - \frac{\epsilon\mu}{2}\right) + \frac{\epsilon}{2} \times \left(z^{D*} - f(z^{L*}, z^{D*})\right) \tag{57}$$

Because ϵ is bounded above by $1 - z_B^M$, and z^{D*} is bounded below by z_B^M (h_1 is increasing in z^{D*}, for a fixed l^*), evaluating the second term at these bounds shows that the term is greater than zero. Then, we can sign h_1 by comparing the first and third terms:

$$h_1 > \frac{\epsilon}{2} \times \left(z_B^M - \frac{\mu(z_B^M + z^{L*})}{2} - l_B \cdot \frac{\mu}{2}\right) \tag{58}$$

Using the fact that $z^{L*} = \frac{\mu z^{D*}}{2-\mu}$ from (52), and $l_B = \frac{\rho \mathsf{Pr}(z \leq -z_B^M)(z_B^M - f(z_B^M, z_B^L))}{z_B^M - f(z_B^M, z_B^M)}$, I simplify (58) to:

$$h_1 > \frac{\epsilon}{2} \times \left(z_B^M - \frac{\mu(1 - z_B^M)}{2(1-\mu)}\right) \tag{59}$$

Then, by rearranging (59), $h_1 > 0 \iff z_B^M \geq \frac{\mu}{2-\mu}$, which is a necessary condition of any equilibrium where market orders are used. Thus, if $z^{M*} > z_B^M$, then $h_1 > 0$. This implies that dark orders are preferred to limit orders if $z^{D*} = z_B^M$, and thus any z^{D*} that solves (51) when $z^{M*} > z_B^M$, must be such that $z^{D*} < z_B^M$. Hence, $z^{M*} \geq z_B^M \geq z^{D*} \geq 0$. Finally, because $z^{D*} \leq z_B^M$, condition (52) yields $z^{L*} \leq z_B^L$.

Step 2. To prove that in any equilibrium where $0 \leq z^{L*} \leq z^{D*} \leq z^{M*} \leq 1$, it must be that $\lambda < \lambda^*$, note that the following must hold in equilibrium:

$$\mathsf{E}[\delta \mid z^{D*} \leq z < z^{M*}] = (1-2\lambda)\mathsf{E}[\delta \mid z \geq z^{M*}] \iff \lambda = \frac{1 - z^{D*}}{2(1+z^{M*})} \quad < \frac{1 - z_B^L}{2(1+z_B^M)} = \lambda^* \tag{60}$$

which follows from the result in step 1 that $z^{D*} > z_B^L$, and $z^{M*} > z_B^M$.

Existence and Uniqueness

For an equilibrium to exist where thresholds satisfy $0 \leq z^{L*} \leq z^{D*} \leq z^{M*} \leq 1$, it must be true that $(1-2\lambda) \times \mathsf{E}[\delta \mid z \geq z^{M*}] > \mathsf{E}[\delta \mid z^{L*} \leq z < z^{D*}]$, by Lemma 2. Moreover, the first part of the proof restricts the ranges of the threshold values to $0 < z^L \leq z_B^L \leq z^{D*} \leq z_B^M \leq z^M \leq 1$.

Step 1: Existence and Uniqueness of $z^{L*}(z^M)$

By rearranging (48), I obtain $z^{L*} = \frac{\mu \cdot z^D}{2-\mu} \in [0, z_B^L]$, which exists and is unique for all $z^D \in [z_B^L, z_B^M]$.

Step 2: Existence and Uniqueness of $l^*(z^M)$

Here, I show that a unique $l^* \in [\rho \mathsf{Pr}(z \leq -z^M), 1]$ solves (46) for all $z^M \in [z_B^M, 1]$. First, evaluating $\Delta^l_{\lambda<\lambda^*}$ at the lower bound $l^* \in [\rho \mathsf{Pr}(z \leq -z^M), 1]$, we have that $\Delta^l_{\lambda<\lambda^*} > 0$.

$$\begin{aligned} \Delta^l_{\lambda<\lambda^*} &= z^M - \mathsf{E}[\delta \mid z \geq z^M] - \rho \mathsf{Pr}(z \leq -z^M) \times (z^M - (1-2\lambda)\mathsf{E}[\delta \mid z \geq z^{M*}]) \\ &> z_B^M - \mathsf{E}[\delta \mid z \geq z_B^M] - \rho \mathsf{Pr}(z \leq -z_B^M) \times \left(z_B^M - \frac{\mu(z_B^M + z_B^L)}{2} \right) = 0 \end{aligned} \tag{61}$$

where the final step follows from the proof of Theorem 1. To show that (61) holds for all $z^M > z_B^M$, note first that (61) is decreasing in μ and ρ, so evaluate (61) at $\mu = \rho = 1$. Then, differentiating by z^M:

$$\frac{\partial \Delta^l_{\lambda<\lambda^*}}{\partial z^M} = \frac{z^M(1+2\lambda)}{2} > 0 \tag{62}$$

And hence (61) is negative for all $z^M \in [z_B^M, 1]$. Next, evaluate (46) at the upper bound, $l^* = 1$:

$$\Delta^l_{\lambda<\lambda^*} = z^M - \frac{\mu(1+z^M)}{2} - 1 \times \left(z^M - (1-2\lambda) \times \frac{\mu(1+z^M)}{2} \right) < 0 \tag{63}$$

Then, by the intermediate value theorem, l^* exists. Lastly, differentiating $\Delta^l_{\lambda<\lambda^*}$ by l^*, we have:

$$\frac{\partial \Delta^l_{\lambda<\lambda^*}}{\partial l^*} = -\left(z^M - (1-2\lambda)\mathsf{E}[\delta \mid \mathsf{O} = \mathsf{MB}^*] \right) < 0 \tag{64}$$

which holds for any $z^M \geq z_B^M$ from the proof of Theorem 1. Thus, l^* is unique for all $z^M \in [z_B^M, 1]$.

Step 3: Existence and Uniqueness of $z^{D*}(z^M)$

I now show that there exists a unique $z^{D*} \in [z_B^L, z_B^M]$ that solves (49) for all $z^M \in [z_B^M, 1]$, and some subset of $\lambda \in (0, \lambda^*)$. First, solve (49) for z^D:

$$\Delta^D_{\lambda<\lambda^*} = 0 \iff z^{D*} = (1-2\lambda)(1+z^M) - z^M \tag{65}$$

Thus, given λ and z^M, if z^{D*} exists, it is unique. Solving for $\lambda = \frac{1-z^{D*}}{2(1+z^M)}$ yields the possible interval of λ for which a $z^{D*} \in [z_B^L, z_B^M]$ can exist. Because λ is decreasing in z^M and z^{D*}, we evaluate our expression for λ from (65) at the upper bounds of z^M and z^{D*} to obtain the lower bound of this interval, and conversely for the upper bound. This yields the interval $\left[\frac{1-z_B^M}{4}, \frac{1-z_B^L}{2(1+z_B^M)}\right] \subseteq (0, \lambda^*]$.

Step 4: Existence and Uniqueness of z^{M*}

Lastly, I show that there exists a unique $z^{M*} \in [z^M_B, 1]$ that solves (47). Recall that $z^{D*} = (1-2\lambda)\mathsf{E}[\delta \mid z \geq z^M] - z^M$. First, note that any solution z^{M*} to (47) must also satisfy the requirement that $z^{D*}(z^{M*})$ crosses zero from below. That is, for any $z < z^{D*}$, $\Delta^M_{\lambda<\lambda^*} < 0$. Then, because z^{D*} is decreasing in z^M for any $\lambda \in (0, \lambda^*)$ (by inspection), the value z^{M*} that solves $\Delta^M_{\lambda<\lambda^*} = 0$ must cross zero from above.

Substitute $z^{D*} = (1-2\lambda)\mathsf{E}[\delta \mid z \geq z^M] - z^M$ and $l^* = \frac{2z^M - \mu(1+z^M)}{2z^M - (1-2\lambda)\mu(1+z^M)}$ into (47):

$$\begin{aligned}\Delta^M_{\lambda<\lambda^*} = &\left(\frac{2z^M - \mu(1+z^M)}{2z^M - (1-2\lambda)\mu(1+z^M)}\right) \cdot \left((1-2\lambda)\mathsf{E}[\delta \mid z \geq z^M] - z^M - \frac{\mu z^M}{2}\right) \\ &- \rho \mathsf{Pr}(z \leq -z^M) \times \frac{2(1-\mu)}{2-\mu}\left((1-2\lambda)\mathsf{E}[\delta \mid z \geq z^M] - z^M\right) \end{aligned} \tag{66}$$

Expanding (66) yields that $\Delta^M_{\lambda<\lambda^*} = 0$ is cubic polynomial in z^M, with a negative coefficient on ${z^M}^3$. Thus, there exist at most 2 solutions to $\Delta^M_{\lambda<\lambda^*} = 0$ that cross zero from above. I show that at most one solution is admissible, by proving that the left-most root is negative. Evaluate $\Delta^M_{\lambda<\lambda^*}$ at $z^M = 0$:

$$\Delta^M_{\lambda<\lambda^*}(z^M = 0) = -\frac{(1-2\lambda)\mu(2(1-\mu)(2-\rho(1-2\lambda)) + \mu^2)}{4(2-\mu)} < 0 \tag{67}$$

Then, because $\Delta^M_{\lambda<\lambda^*}(z^M = 0)$ is negative, it must be that the left-most root is negative, which cannot be an admissible solution to $\Delta^M_{\lambda<\lambda^*} = 0$. Hence, any z^M that solves $\Delta^M_{\lambda<\lambda^*}$ given $\lambda \in (0, \lambda^*)$ is unique.

Finally, I show that there exists a non-empty interval $\lambda \in [\lambda_1, \lambda_2] \subset (0, \lambda^*)$ in which a unique z^{M*} exists for every λ. First, I evaluate $\Delta^M_{\lambda<\lambda^*}$ at the endpoint $z^M = 1$, and solve for λ, which yields a unique $\lambda = \frac{1-\mu}{2(2-\mu)} \equiv \lambda_1 \in (0, \lambda^*)$. Then, differentiate $\Delta^M_{\lambda<\lambda^*}(z^M)$ with respect to λ:

$$\begin{aligned}\frac{\partial \Delta^M_{\lambda<\lambda^*}}{\partial \lambda} &= \frac{\partial l^*}{\partial \lambda} \cdot \left(z^{D*} - \frac{(1-2\lambda)\mu(1+z^M)}{2}\right) - (1+z^M)\left(l^*(2-\mu) - \frac{\rho(1-z^M)}{2}\left(\frac{1-\mu}{2-\mu}\right)\right) \\ &< \frac{\partial l^*}{\partial \lambda} \cdot \left(z^{D*} - \frac{(1-2\lambda)\mu(1+z^M)}{2}\right) - (1+z^M)\left(l^* - \frac{\rho(1-z^M)}{2}\right) < 0 \end{aligned} \tag{68}$$

where (68) is negative by the fact that $l^* > \frac{\rho(1-z^M)}{2}$ and $\frac{\partial l^*}{\partial \lambda} < 0$. To see the latter, I compute $\frac{\partial l^*}{\partial \lambda}$:

$$\frac{\partial l^*}{\partial \lambda} = -\frac{\lambda(1+z^M)\mu(2z^M - \mu(1+z^M))}{(2z^M - (1-2\lambda)\mu(1+z^M))^2} < 0 \tag{69}$$

That $\Delta^M_{\lambda<\lambda^*}$ is decreasing in λ for all z^M implies that λ_1 is the lower bound for an equilibrium, and further, that z^{M*} exists and is unique for all $\lambda \in [\lambda_1, \lambda_2]$, for some upper bound, λ_2. To complete the proof, I show that $\lambda_2 < \lambda^*$. We know that a unique λ_2 exists, because the cubic nature of $\Delta^M_{\lambda<\lambda^*}$ implies a local maximum to the left of any z^{M*} that is always above zero until z^{M*} jointly satisfies $\Delta^M_{\lambda<\lambda^*} = 0$ and $\frac{\partial \Delta^M_{\lambda<\lambda^*}}{\partial z^M} = 0$ for some $\lambda > \lambda_1$. Then, evaluate $\Delta^M_{\lambda<\lambda^*}$ at $\lambda = \frac{2-\mu(1+z^M)}{2(1+z^M)(2-\mu)} \in (\lambda_1, \lambda^*)$, yielding:

$$\Delta^M_{\lambda<\lambda^*} = -\frac{2{z^M}^2(1-\mu)^2\rho(1-z^M)\mu}{(2-\mu)^3} < 0 \tag{70}$$

This implies that $\lambda_2 \in \left(\lambda_1, \frac{2-\mu(1+z^M)}{2(1+z^M)(2-\mu)}\right)$. Hence, a unique z^{M*} that solves $\Delta^M_{\lambda<\lambda^*}$ for all $\lambda \in [\lambda_1, \lambda_2]$. ■

Proof (Theorem 3). Similar to the proof of Theorem 2, I first show that for any equilibrium satisfying Definitions (1)-(2), $\lambda \in [\lambda^*, 1/2)$ if and only if the equilibrium threshold values satisfy $0 \le z^{D*} \le z^L_B \le z^{L*} \le z^M_B \le z^{M*} \le 1$. I proceed to show that there exist unique z^{D*}, z^{L*} and l^* for all z^M, and that z^{M*} uniquely exists for for all $\lambda \in [\lambda^*, 1/2)$. Throughout, I recall the conditions from (23)-(26).

$$\Delta^M_{\lambda>\lambda^*} \equiv z^M - \mathsf{E}[\delta \mid z \ge z^M] - \rho \cdot \mathsf{Pr}(z \le -z^M) \times \left(z^M - \mathsf{E}[\delta \mid z^L \le z < z^M]\right) \tag{71}$$

$$\Delta^l_{\lambda>\lambda^*} \equiv \rho \mathsf{Pr}(z \le -z^M) \times \left(z^L - \mathsf{E}[\delta \mid z^L \le z < z^M]\right) - l \cdot \left(z^L - (1-2\lambda)\mathsf{E}[\delta \mid z \ge z^M]\right) \tag{72}$$

$$\Delta^D_{\lambda>\lambda^*} \equiv z^D - (1-2\lambda) \times \mathsf{E}[\delta \mid z \ge z^M] \tag{73}$$

$$\Delta^L_{\lambda>\lambda^*} \equiv (1-2\lambda)\mathsf{E}[\delta \mid z \ge z^M] - \mathsf{E}[\delta \mid z^D \le z < z^L] \tag{74}$$

Marginal Valuation Threshold Ranking

Let equilibrium threshold values satisfy $0 \le z^{D*} \le z^{L*} \le z^{M*} \le 1$. Then, in an equilibrium where all order types are used, I show that equilibrium threshold values (indicated by $*$) must also satisfy $0 \le z^{D*} \le z^L_B \le z^{L*} \le z^M_B \le z^{M*} \le 1$. Consider the equilibrium conditions derived from (23)-(26):

$$z^{M*} - f(z^{M*}, 1) - \mathsf{Pr}(-z \le z^{M*}) \times (z^{M*} - f(z^L, z^{M*})) = 0 \tag{75}$$

$$\rho \mathsf{Pr}(-z \le z^{M*}) \times (z^{L*} - f(z^{L*}, z^{M*})) - l^* \times (z^{L*} - f(z^{D*}, z^{L*})) = 0 \tag{76}$$

$$z^{D*} - f(z^{L*}, z^{D*}) = 0 \tag{77}$$

Let $z^{M*} > z^M_B$ and $z^{L*} < z^L_B$. Then it must be true that $\mathsf{Pr}(-z \le z^{M*}) < \rho \mathsf{Pr}(z \le -z^M_B)$, and that $z^{L*} > \frac{\mu z^{M*}}{2-\mu} > \frac{\mu z^M_B}{2-\mu} > z^L_B$, a contradiction. Instead, suppose then that $z^{L*} > z^L_B$. Then, $\pi_L(z^{M*}) < \pi_L(z^M_B)$, implying that the z^{M*} that solves (75) must be such that $z^{M*} < z^M_B$, a contradiction. Hence, $z^{M*} \le z^M_B$.

Then, let $z^{M*} \le z^M_B$ and $z^{L*} < z^L_B$. Because $f(z^{L*}, z^{M*}) < f(z^L_B, z^M_B)$, it must be that $\pi_L(z^{M*}) >$

$\pi_L(z_B^M) \Rightarrow z^{M*} > z_B^M$, a contradiction. Thus, $z^{L*} \geq z_B^L$. Finally, it must be the case that $z^{D*} \leq z_B^L$, which obtains from solving (77) for z^{D*}: $z^{D*} = \frac{\mu z^{L*}}{2-\mu} \leq \frac{\mu z_B^M}{2-\mu} = z_B^L$.

To prove that thresholds $0 \leq z^{D*} \leq z^{L*} \leq z^{M*} \leq 1$ can only form an equilibrium when $\lambda > \lambda^*$, rearrange $\mathsf{E}[\delta \mid z^D \leq z < z^L] = (1-2\lambda) \times \mathsf{E}[\delta \mid z \geq z^M]$ to isolate for λ:

$$\lambda = \frac{1 + z^{M*} - z^{L*} - z^{D*}}{2(1+z^{M*})} \geq \frac{1 - z^{D*}}{2(1+z^{M*})} \geq \frac{1 - z_B^L}{2(1+z_B^M)} = \lambda^* \tag{78}$$

which we arrive at by the fact that $z^{D*} \leq z_B^L$.

Existence and Uniqueness

For an equilibrium to exist where threshold values satisfy $0 < z^D < z^L \leq z^M < 1$, it must be true that $(1-2\lambda) \times \mathsf{E}[\delta \mid z \geq z^M] < \mathsf{E}[\delta \mid z^L \leq z < z^M]$, by Lemma 2. The argument above restricts the ranges of the threshold values to $0 \leq z^D \leq z_B^L \leq z^L \leq z^M \leq z_B^M < 1$.

Step 1: Existence and Uniqueness of $l^*(z^L)$

To show that a unique $l^* \in [0, \rho\mathsf{Pr}(z \leq -z^M)]$ exists that solves (72) for all $z^M \in [z_B^L, z^M]$ and $z^M \in [z^L, z_B^M]$, rearrange (72):

$$l^* = \frac{\rho\mathsf{Pr}(z \leq -z^M) \times (z^L - \mathsf{E}[\delta \mid z^L \leq z < z^M])}{z^L - (1-2\lambda)\mathsf{E}[\delta \mid z \geq z^M]} < \rho\mathsf{Pr}(z \leq -z^M) \tag{79}$$

Thus, a unique $l^* < \rho\mathsf{Pr}(z \leq -z^M)$ exists, as the denominator of (79) is positive when (73) is satisfied. Moreover, $(1-2\lambda) \times \mathsf{E}[\delta \mid z \geq z^M] < \mathsf{E}[\delta \mid z^L \leq z < z^M]$ implies that the inequality is strict.

Step 2: Existence and Uniqueness of $z^{D*}(z^L)$

I now show that there exists a unique $z^D \in [0, z^L]$ that solves (73) for all $z^L \in [z_B^L, z^M]$, and $z^M \in [z^L, z_B^M]$. By combining (73) and (74), we can solve to obtain $z^{D*} = \frac{\mu z^L}{2-\mu}$, which exists uniquely for all $z^L \in [z_B^L, z^M]$.

Step 3: Existence and Uniqueness of $z^{M*}(z^L)$

To show that there exists a unique $z^{M*} \in [z^L, z_B^M]$ that solves (46) for all $z^L \in [z_B^L, z^M]$, evaluate $\Delta^M_{\lambda > \lambda^*}(z^M)$ at the endpoints:

$$\Delta^M_{\lambda>\lambda^*}(z^M = z^L) = \left(z^L - \tfrac{\mu(1+z^L)}{2}\right) - \rho\tfrac{(1-z^L)}{2} \times (z^L - \mu z^L) \tag{80}$$

$$\Delta^M_{\lambda>\lambda^*}(z^M = z^M_B) = z^M_B - \tfrac{\mu(1+z^M_B)}{2} - \rho\tfrac{(1-z^M_B)}{2} \times \left(z^M_B - \tfrac{\mu(z^M_B+z^L)}{2}\right) \tag{81}$$

To see that (81) is non-negative, consider $z^L = z^L_B$. Then, (81) is zero from the proof of Theorem 1. Thus, for any $z^L > z^L_B$, it must be that $\Delta^M_{\lambda>\lambda^*}(z^M = z^M_B) > 0$, as it is increasing in z^L.

Consider Equation (80). If $z^L = z^L_B$, then $\Delta^M_{\lambda>\lambda^*} < 0$ because $\Delta^M_{\lambda>\lambda^*}(z^M = z^M_B, z^L = z^L_B) = 0$, and $\frac{\partial \Delta^M_{\lambda>\lambda^*}}{\partial z^M} > 0$. Then, coupled with the fact that (81) is non-negative, we have that there exists a $z^{\hat{M}} = z^L$ such that $\Delta^M_{\lambda>\lambda^*} = 0$. Thus, for all $z^M < \widehat{z^M}$, equation (80) is negative. Therefore, $z^{M*} < \widehat{z^M}$ exists. Then, the uniqueness of $z^{M*} < \widehat{z^M}$ follows from $\Delta^M_{\lambda>\lambda^*}$ increasing in z^M:

$$\frac{\partial \Delta^M_{\lambda>\lambda^*}}{\partial z^M} = 1 - \frac{\mu}{2} + \frac{\rho}{2} \times (z^M - \mathsf{E}[\delta \mid z^L \le z < z^M]) - \rho \mathsf{Pr}(z \le -z^M) \times \left(1 - \tfrac{\mu}{2}\right) > 0 \tag{82}$$

For $z^{M*} > \widehat{z^M}$, I show that $z^{M*} = z^L$ is the unique value. Suppose that $z^{M*} = z^L$. z^{D*} and λ^* follow as in steps 2 and 3. To show that a unique l^* exists, we can combine equilibrium conditions (71) and (72), and substitute $\mathsf{E}[\delta \mid \mathsf{O} = \mathsf{DB}^*] = \frac{\mu z^L}{2-\mu}$ to achieve:

$$\Delta^M_{\lambda>\lambda^*}(z^M = z^L) = \left(z^L - \tfrac{\mu(1+z^L)}{2}\right) - l^* \times \left(z^L - \tfrac{\mu z^L}{2-\mu}\right) \tag{83}$$

By inspection, any $l^* \in [0, \rho \mathsf{Pr}(z \le -z^M)]$ that solves (83), $\forall z^L \in [\widehat{z^M}, z^M_B]$ is unique. Hence, if $z^{M*} > \widehat{z^M}$, a unique equilibrium exists where $z^{M*} = z^{L*}$. Thus, z^{M*} exists and is unique for all $z^L \in [z^L_B, z^M_B]$.

Step 4: Existence and Uniqueness of z^{L*}

To show there exists a unique $z^{L*} \in [z^L_B, z^{M*}]$ that solves (74) for all $\lambda \ge \lambda^*$, I evaluate (74) at z^{D*}:

$$\Delta^L_{\lambda>\lambda^*}(z^L) = z^L - (1-2\lambda)(1+z^{M*}) \cdot (1 - \tfrac{\mu}{2}) \tag{84}$$

Thus, the existence of z^{L*} depends on λ. To determine the bounds on λ, first obtain $\frac{\partial \Delta^L_{\lambda>\lambda^*}(z^L)}{\partial z^L}$:

$$\frac{\partial \Delta^L_{\lambda>\lambda^*}(z^L)}{\partial z^L} = 1 - (1-2\lambda)\left(1 - \frac{\mu}{2}\right) \cdot \frac{\partial z^M}{\partial z^L} > 0 \tag{85}$$

which is positive by the fact that $\Delta^M_{\lambda>\lambda^*}$ is increasing in z^M and z^L: if z^M increases, then z^L must decrease for $\Delta^M_{\lambda>\lambda^*} = 0$. Hence, $\frac{\partial z^M}{\partial z^L} < 0$. $\Delta^L_{\lambda>\lambda^*}(z^L)$ is increasing in z^L. Now, evaluate $z^{L*} = z^L_B$, its lower bound. Doing so implies that $z^{M*} = z^M_B$, as (71) becomes as in Theorem 1. Solving for λ, I obtain

$\lambda = \frac{1}{2} - \frac{z_B^L}{(2-\mu)(1+z_B^M)} \equiv \lambda_4 < \frac{1}{2}$. Hence, $\lambda \leq \lambda_4$. Moreover, at $\lambda = \lambda_4$ it must be thatt $l^* = 0$, as $z^{D*} - \mathsf{E}[\delta \mid z^{D*} \leq z \leq z_B^L] < 0$ for all $z^{D*} < z_B^M$. Now evaluate $z^L = z^M$, the upper bound, and solve for λ to obtain $\lambda = \frac{2-\mu(1+z^M)}{2(2-\mu)(1+z^M)} \equiv \lambda_3 > \lambda^*$. Thus $z^{L*} = z^M \in [z_B^L, z_B^M]$ at $\lambda = \lambda_3$.

Finally, to characterize z^{L*} for $\lambda \in [\lambda^*, \lambda_3)$, consider some $\tilde{\lambda} \in [\lambda^*, \lambda_3)$. Then by (85), $z^L(\tilde{\lambda}) > z^{L*}(\lambda_3) = z^{M*}(\lambda_3) > z^{M*}(\tilde{\lambda})$, $\Rightarrow\Leftarrow$. Now, let $z^{L*} = z^{M*}$. This implies that $\Delta^M_{\lambda > \lambda^*} > 0$ and $\Delta^l_{\lambda > \lambda^*} < 0$ for all $\lambda \in [\lambda^*, \lambda_3)$, and hence, any investor with valuation $z \geq z^{M*}$ prefers market orders to limit orders, and with $z < z^{M*}$ prefers dark orders to limit orders. The only check that remains is to show that z^{M*} forms an equilibrium such that $\Delta^M_{\lambda > \lambda^*} - \Delta^l_{\lambda > \lambda^*} = 0$ (i.e., investors are indifferent to market orders and dark orders at $z^{L*} = z^{M*}$). We then have the condition:

$$\Delta^M(z^{M*}) = z^{M*} - \frac{(1 + z^{M*})\mu}{2} - l^* \left(z^{M*} - \frac{\mu(z^{M*} + z^{M*})}{2} \right) = 0 \tag{86}$$

which, because $z^L = z^{M*} > z_B^L$, holds for a (unique) $l^* \in (0, \rho \mathsf{Pr}(z \leq -z^{M*})$. Thus, a unique equilibrium exists for all $\lambda \in [\lambda^*, \lambda_4)$ such that $z^{L*} = z^{M*}$. ■

A.4 Proofs: Propositions

Proof (Proposition 1). Let $\lambda \in (0, 1)$. Suppose that there exists an equilibrium in threshold strategies that satisfies Definitions 1-2, where displayed limit prices bid^*, ask^* satisfy (6)-(7), $l^* > 0$, and $z^M < 1$. If $l^* > 0$, then with some probability the liquidity pool in any period is non-empty. Since liquidity at the pool is displayed, an investor that arrives at period t either observes the liquidity pool as full or empty. If the liquidity pool is full, then the prices offered at the pool improve upon the lit market by $\lambda(\mathsf{ask}_t - \mathsf{bid}_t)$. Thus, as prices are always better at the liquidity pool, all investors strictly prefer to submit orders to the liquidity pool when the pool is full, and thus $z^M = 1$. $\Rightarrow\Leftarrow$. Hence, it must be that no such equilibrium exists. ■

Proof (Proposition 2). The quoted spread measure, $\mathsf{ask} - \mathsf{bid} = 2\mathsf{E}[\delta \mid z \geq z^{M*}]$, is increasing in z^{M*} only. Thus, by Theorem 2, if $\lambda \in [\lambda_1, \lambda_2]$, then $z^{M*} \geq z_B^M \Rightarrow \mathsf{E}[\delta \mid z \geq z^{M*}] \geq \mathsf{E}[\delta \mid z \geq z_B^M]$. Similarly via Theorem 3, if $\lambda \in [\lambda^*, \lambda_4]$, then $z^{M*} \leq z_B^M \Rightarrow \mathsf{E}[\delta \mid z \geq z^{M*}] \leq \mathsf{E}[\delta \mid z \geq z_B^M]$. Moreover, the price impact is equal to the half-spread, implying identical qualitative results. ■

Proof (Proposition 3). Let $\lambda \in [\lambda_1, \lambda_2]$. Lit market volume is given by $2\mathsf{Pr}(\mathsf{O} = \mathsf{MB}^*) = 2\mathsf{Pr}(z \geq z^{M*})$, which is decreasing in z^{M*}. Thus, by Theorem 2, if $\lambda \in [\lambda_1, \lambda_2]$, then $z^{M*} \geq z_B^M$, and $\mathsf{Pr}(z \geq z^{M*}) \leq \mathsf{Pr}(z \geq z_B^M)$. Total volume is given by $TV_{\lambda_1} = 2(\mathsf{Pr}(\mathsf{O} = \mathsf{MB}^*) + l^*\mathsf{Pr}(\mathsf{O} = \mathsf{DB}^*)) = 2(\mathsf{Pr}(z \geq

$z^{M*}) + l^*\mathsf{Pr}(z^{D*} \le z \le z^{M*}))$. Evaluating at $\lambda = \lambda_1$, $z^{M*} = 1$. Then, taking the difference between the benchmark level of total volume, $TV_B = 2\mathsf{Pr}(z \ge z_B^M)$, and TV_{λ_1}, we arrive at:

$$TV_B - TV_{\lambda_1} = \frac{2 - \mu - (2 - 3\mu + \mu^2)\rho - \sqrt{(1-\mu)^2\rho^2 - (4 - 10\mu + 6\mu^2)\rho + (2-\mu)^2}}{4\rho(1-\mu)} \tag{87}$$

which is non-negative for all $(\mu, \rho) \in (0,1)^2$ by graphical inspection (see Figure 5). Hence, $TV_\lambda \le TV_B$ for λ_1. Because total volume for $\lambda \in [\lambda_1, \lambda_2]$ is continuous in λ, z^{M*}, z^{D*}, it must be that total volume is lower than the benchmark value for all $\lambda \in [\lambda_1, \hat{\lambda}]$, for some $\hat{\lambda} \in [\lambda_1, \lambda_2]$.

If $\lambda \in [\lambda^*, \lambda_4]$, then $z^{M*} \le z_B^M$, and $\mathsf{Pr}(z \ge z^{M*}) \ge \mathsf{Pr}(z \ge z_B^M)$ (via Theorem 3). It directly follows that if lit market volume is higher in $\lambda \in [\lambda^*, \lambda_4]$ than in the lit-only equilibrium, then so is total volume.

Finally, I compare OS to the lit-only equilibrium values. If $\lambda \in [\lambda_1, \lambda^*]$, $OS = 2(1 - z^{L*}) \ge 2(1 - z_B^L)$, as $z^{L*} \le z_B^L$; for $\lambda \in [\lambda^*, \lambda_4]$, $OS = 2(1 - z^{D*}) \ge 2(1 - z_B^L)$ by the fact that $z^{D*} \le z_B^L$. Thus, for any price improvement with an active dark pool, market participation (weakly) increases. ■

Proof (Proposition 4). This proof completes in two parts: i) I show that W in (29) is lower for $\lambda \in [\lambda_1, \lambda_2]$ than in the lit-only case, W_B; ii) I show the reverse is true on $\lambda \in [\lambda^*, \lambda_4]$. First, compute W_B:

$$W_B = \frac{(1-\mu)}{4} \cdot \left(1 - {z_B^M}^2 + \rho\mathsf{Pr}(z \le -z_B^M)({z_B^M}^2 - {z_B^L}^2)\right) = \frac{(1-\mu)}{4}\left(1 - \frac{\mu z_B^M}{2-\mu}\right) \tag{88}$$

which I obtain by substituting the equilibrium value $\rho\mathsf{Pr}(\mathsf{O} = \mathsf{MS}_B) = \frac{2z_B^M - \mu(1+z_B^M)}{2z_B^M + \mu(z_B^M + z_B^L)}$, and simplifying.

Now let $\lambda \in [\lambda_1, \lambda_2]$. I compute W in (29) by using the equilibrium values for $\rho\mathsf{Pr}(z \le -z^{M*})$ from (47) and l^* from (46), and simplify to obtain the second equality:

$$W = \frac{(1-\mu)}{4} \cdot \left(1 - z^{M*2} + l^*(z^{M*2} - z^{D*2}) + \rho\frac{(1 - z^{M*})}{2}(z^{D*2} - z^{L*2})\right) = \frac{(1-\mu)}{4}\left(1 - \frac{\mu z^{M*}}{2-\mu}\right) \tag{89}$$

Hence, $W_B \ge W(z^{M*}) \iff z^{M*} \ge z_B^M$, which is true in equilibrium for all $\lambda \in [\lambda_1, \lambda_2]$. For any other $\lambda \in (0, \lambda^*)$, the equilibrium is as in Theorem 1, implying $z^{M*} = z_B^M \Rightarrow W(z^{M*}) = W_B$

Similarly, for the values under Theorem 3 where $\lambda \ge \lambda^*$, I compute W from (29). Using the equilibrium values for l^* from (72) and $\rho\mathsf{Pr}(\mathsf{O} = \mathsf{MS}^*)$ from (71) yields the simplification:

$$W = \frac{(1-\mu)}{4} \cdot \left(1 - z^{M*2} + \rho\frac{(1 - z^{M*})}{2}(z^{M*2} - z^{L*2}) + l^*(z^{L*2} - z^{D*2})\right) = \frac{(1-\mu)}{4}\left(1 - \frac{\mu z^{M*}}{2-\mu}\right) \tag{90}$$

Hence, $W(z^{M*}) \geq W_B \iff z^{M*} \geq z_B^M$, which is true for all $\lambda \in [\lambda^*, \lambda_3)$. For any other $\lambda \in [\lambda^*, 1)$, the equilibrium is as in Theorem 1, where $W(z^{M*}) = W_B$. Finally, (89) and (90) provide that W is decreasing in z^{M*}, and thus welfare comoves negatively with the quoted spread, $\mathsf{ask} - \mathsf{bid} = \mu(1+z^{M*})$, and positively with lit market volume, $(1 - z^{M*})$. ■

Proof (Proposition 5). I begin by proving that that the expected conditional price impact is more efficient than in the benchmark equilibrium by showing that $PD(l^* = 1)$ is lower than PD_B. For the expected conditional price impact, the fill rate of a dark order is irrelevant, so we set $l_{t-1}^* = 1$ in (31). Evaluating (31) at the benchmark values, we have $PD_B = 1 - \mu^2 \cdot \left(1 - {z_B^L}^2\right)$. Similarly, evaluating at the equilibrium values obtained from Theorem 2 and 3:

$$PD(\lambda \in [\lambda_1, \lambda_2]; l^* = 1) = 1 - \mu^2 \cdot \left(1 - {z^{L*}}^2\right) \tag{91}$$

$$PD(\lambda \in [\lambda^*, \lambda_4]; l^* = 1) = 1 - \mu^2 \cdot \left(1 - {z^{D*}}^2\right) \tag{92}$$

where $PD(\lambda \in [\lambda_1, \lambda_2]; l^* = 1) < PD_B$ by the fact that $z^{L*} \leq z_B^L$, and thus, the conditional price impact is more efficient. Similarly $PD(\lambda \in [\lambda^*, \lambda_4]; l^* = 1) < PD_B$ by the fact that $z^{D*} \leq z_B^L$.

To show that price efficiency (the unconditional price impact) is less efficient in all cases, first consider $\lambda \in [\lambda_1, \lambda_2]$. I can simplify (31) by substituting equilibrium values for l^* and z^{L*} to obtain:

$$PD(\lambda \in [\lambda_1, \lambda_2]) = 1 - \mu^2 \cdot \left(1 - {z^{M*}}^2 + \frac{((2-\mu)z^{M*} - \mu)({z^{M*}}^2 - {z^{D*}}^2)}{(2-\mu)z^{M*} - \mu z^{D*}} + \left(1 - \left(\frac{\mu}{2-\mu}\right)^2\right) {z^{D*}}^2\right) \tag{93}$$

First, let z^{D*} be independent of z^{M*}. Then, (93) is increasing in z^{M*} for any fixed z^{D*}:

$$\frac{\partial PD(\lambda \in [\lambda_1, \lambda_2])}{\partial z^{M*}} = \frac{\mu^3((2-\mu)({z^{M*}}^2 + {z^{D*}}^2) - 2\mu z^{M*} z^{D*})(1 - z^{D*})}{(2-\mu)^2} > 0 \tag{94}$$

Thus, $PD(\lambda \in [\lambda_1, \lambda_2])$ minimizes at $z^{M*} = z_B^M \; \forall z^{D*}$. Evaluate $PD(\lambda \in [\lambda_1, \lambda_2]; z^{M*} = z_B^M) - PD_B$:

$$PD(\lambda \in [\lambda_1, \lambda_2]; z^{M*} = z_B^M) - PD_B = \frac{({z_B^M}^2 - {z^{D*}}^2)\mu^3(4(1-\mu)(1 - z^{D*}) + \mu^2(1 + z_B^M) - 2\mu z_B^M)}{(2-\mu)^2(2z_B^M - \mu(z^{D*} + z_B^M))} \tag{95}$$

which is non-negative for all z^{D*} if:

$$4(1-\mu)(1-z_B^M)+\mu^2(1+z_B^M)-2\mu z_B^M \geq 0 \tag{96}$$

By the fact that $z^{D*} \leq z_B^M$. Then, evaluating (96) at z_B^M in (45), the expression is non-negative for all $(\mu, \rho) \in (0,1)^2$ by graphical inspection (see Figure 5). Now, let $\lambda \in [\lambda^*, 1)$. Simplifying (31), I obtain:

$$PD(\lambda \in [\lambda^*, \lambda_4]) = 1-\mu^2 \cdot \left(1 - z^{L*2} + \frac{\rho(1-z^{M*})}{2} \cdot \left(z^{L*} - \frac{\mu z^{M*}}{2-\mu}\right) z^{L*}\right) \tag{97}$$

To show that $PD(\lambda \in [\lambda^*, \lambda_4]) \geq PD_B$, first note that (97) is decreasing in z^{L*}:

$$\frac{\partial PD(\lambda \in [\lambda^*, \lambda_4])}{\partial z^{L*}} = -\frac{z^{M*}(1-z^{M*})\mu + 2(2-\mu)(1+z^{M*})z^{L*})}{2(2-\mu)} \tag{98}$$

Then, evaluating $PD(\lambda \in [\lambda^*, \lambda_4]) - PD_B$ at the lower bound of $z^{L*} = \frac{\mu z_B^M}{2-\mu}$ yields:

$$PD(\lambda \in [\lambda^*, \lambda_4]) - PD_B = \frac{\mu^4 \rho z_B^M (1-z^{M*})(z_B^M - z^{M*})}{2(2-\mu)^2} \tag{99}$$

which is non-negative for all $z^{M*} \in [z^{L*}, z_B^M]$. ■

Proof (Proposition 6). The proof proceeds in two parts. I show that: i) dark volume is greater for some $\lambda \in [\lambda_1, \lambda_2]$ than $\forall \lambda \in [\lambda^*, \lambda_4]$, and; ii) total volume is larger for some $\lambda \in [\lambda^*, \lambda_4]$ than $\forall \lambda \in [\lambda_1, \lambda_2]$.

Step 1. First, suppose that $\lambda = \lambda_1 = \frac{1-\mu}{2(2-\mu)}$. The expression for dark volume simplifies to: $2l^* \mathsf{Pr}(\mathsf{O} = \mathsf{DB}^* \mid \lambda = \lambda_1) = l^* \times (z^{M*}(\lambda = \lambda_1) - z^{D*}(\lambda = \lambda_1)) = l^* \times (1 - z^{D*}(\lambda = \lambda_1))$. Next, compute dark volume for $\lambda \in [\lambda^*, \lambda_4]$, which yields $\mathsf{Pr}(\mathsf{O} = \mathsf{DB}^* \mid \lambda \in [\lambda^*, \lambda_4]) = l^* \times (z^{L*}(\lambda \in [\lambda^*, \lambda_4]) - z^{D*}(\lambda \in [\lambda^*, \lambda_4])) < \frac{\rho(1-z^{M*}(\lambda \in [\lambda^*, \lambda_4])}{2} z_B^M$. The inequality follows from $l^*(\lambda \in [\lambda^*, 1)) < \rho \mathsf{Pr}(\mathsf{O} = \mathsf{MS} \mid \lambda \in [\lambda^*, \lambda_4]) = \frac{\rho(1-z^{M*}(\lambda \in [\lambda^*, \lambda_4]))}{2}$ and $z_B^M > z^{L*}(\lambda \in [\lambda^*, \lambda_4])$.

To see that $(1 - z^{D*}(\lambda = \lambda_1)) \geq (1 - z^{M*}(\lambda \in [\lambda^*, \lambda_4])$, note that in equilibrium, $z^{D*}(\lambda = \lambda_1) = \frac{\mu}{2-\mu}$, as $z^{M*}(\lambda = \lambda_1) = 1$. Then, for any equilibrium where $\lambda \geq \lambda^*$ it must be true that $z^{M*}(\lambda \in [\lambda^*, \lambda_4]) \geq \frac{\mu}{2-\mu}$ for an investor with valuation $z^{M*}(\lambda \in [\lambda^*, \lambda_4])$ to submit a market order and earn non-negative profit. Lastly, to see that $l^*(\lambda = \lambda_1) > \frac{z_B^M}{2}$, note that $l^*(\lambda = \lambda_1) = \frac{2-\mu}{2}$. Rearranging, we have $z_B^M + \mu < 2$, which is true for all $(z_B^M, \mu) \in (0,1)^2$. Hence, a dark pool that selects $\lambda = \lambda_1$ earns greater expected profits than at any $\lambda \in [\lambda^*, \lambda_4]$. It follows from Propositions 2-4 that lit and total volume, and welfare are lower and the quoted spread is wider than in the lit-only equilibrium.

Moreover, λ_1 also admits maximum dark volume for all $\lambda \in [\lambda_1, \lambda_2]$. To see this, we have from the proof of Theorem 2 that l^* is increasing in λ, and that $\Delta^M_{\lambda < \lambda^*} < 0$ at $\lambda = \tilde{\lambda} = \frac{2-\mu(1+z^M)}{2(1+z^M)(2-\mu)}$ for all $z^M \in [z_B^M, 1]$.

Then, in equilibrium, z^{D*} must be larger than the value of z^{D*} evaluated at $\lambda = \tilde{\lambda}$. Hence, if dark volume at λ_1 exceeds dark volume at $\tilde{\lambda}$ for any $z^M \in [z_B^M, 1]$, dark volume is maximal at λ_1. Taking the difference and simplifying, we arrive at the desired result below:

$$2l^*\mathsf{Pr}(\mathsf{O} = \mathsf{DB}^* \mid \lambda = \lambda_1) - 2l^*\mathsf{Pr}(\mathsf{O} = \mathsf{DB}^* \mid \lambda = \tilde{\lambda}) \geq 0 \iff \left(1 - \frac{\mu}{2-\mu}\right)(1 - z^{M*}) > 0 \quad (100)$$

Step 2. The expression for total volume is given by: $TV \equiv 2 \times (\mathsf{Pr}(\mathsf{O} = \mathsf{MB}^*) + l^*\mathsf{Pr}(\mathsf{O} = \mathsf{DB}^*)$. First, I write $TV(\lambda \in [\lambda_1, \lambda_2])$ and simplify in terms of z^{D*} and z^{M*}:

$$TV(\lambda \in [\lambda_1, \lambda_2]) = \frac{1 - z^{M*}}{2} + l^* \cdot \frac{(z^{M*} - z^{D*})}{2} = \frac{1 - z^{M*}}{2} + \frac{((2-\mu)z^{M*} - \mu)(z^{M*} - z^{D*})}{(2(2-\mu)z^{M*} - \mu z^{D*})} \quad (101)$$

Recall from the proof of Theorem 2 that $\lambda_2 < \frac{2-\mu-\mu z^{M*}}{2(2-\mu)(1+z^{M*})} \equiv \tilde{\lambda}$. Then, since (101) is decreasing in z^{D*}, and z^{D*} is decreasing in λ, it must be that $TV(\lambda \in [\lambda_1, \lambda_2]) < TV(\lambda = \tilde{\lambda})$. Hence, I can prove that $TV(\lambda \in [\lambda_1, \lambda_2])$ is lower than for some $\lambda \in [\lambda^*, \lambda_4]$ by proving that $TV(\lambda = \lambda_3) - TV(\lambda = \tilde{\lambda})$ is positive. Evaluate $z^{D*} = (1-2\lambda)(1+z^{M*}) - z^{M*}$ in (101) at $\lambda = \tilde{\lambda}$ to obtain $TV(\lambda = \tilde{\lambda}) = \frac{2-(1+z^{M*})\mu}{4}$, which is decreasing in z^{M*}. Thus, $TV(\lambda = \tilde{\lambda})$ is maximized at the lowest z^{M*}, given by $z^{M*} = z_B^M$ from (45). By simplification, $TV(\lambda = \lambda_3) - TV(\lambda = \tilde{\lambda}; z^{M*} = z_B^M) \geq 0$ if and only if:

$$\mu(2-\mu) + r_1(\mu, \rho) \geq 2\sqrt{4(1-\mu)^2\rho^2 - (16 - 48\mu + 44\mu^2 - 12\mu^3)\rho + (2-\mu)^4)} \quad (102)$$

where $r_1(\mu, \rho) = \sqrt{(1-\mu)^2\rho^2 - (4 + 10\mu - 6\mu^2)\rho + (2-\mu)^2}$. Then, by graphical inspection on $(\mu, \rho) \in (0,1)^2$ (see Figure 5), (102) holds. Thus, $TV(\lambda = \lambda_3) \geq TV(\lambda = \tilde{\lambda}) > TV(\lambda \in [\lambda_1, \lambda_2])$. Thus, the price improvement that maximizes total volume must be in the "large price improvement" interval, which by Propositions 2-4 implies that lit market volume increases, the quoted spread narrows, and investor welfare improves compared to the benchmark equilibrium. ■

A.5 Out-of-Equilibrium Limit Orders and Beliefs

In this paper, I employ the perfect Bayesian equilibrium concept. On-the-equilibrium-path, investors submit limit orders with competitive limit prices. Off-the-equilibrium path, I require an appropriate set of beliefs to ensure that competitive limit prices strategically dominate any off-equilibrium-path deviations in limit price. Intuitively, any limit order posted at a price worse than the competitive equilibrium price is strategically dominated by the competitive price, as the professional liquidity provider reacts to the non-competitive order by undercutting it. For non-competitive limit orders that undercut the competitive price (i.e., a price inside the competitive spread), however, it is not immediate that the competitive price strategically dominates.

Perfect Bayesian equilibrium prescribes that investors and the professional liquidity provider update their beliefs by Bayes rule, whenever possible, but does not place any restrictions on the beliefs of market participants when they encounter an out-of-equilibrium action. To support competitive prices in equilibrium, I assume that if a limit buy order is posted at a price different to the competitive equilibrium bid price bid^*_{t+1}, then market participants hold the following beliefs regarding this investor's private information at period t.

If a limit buy order is posted at a price $\widehat{\mathsf{bid}} < \mathsf{bid}^*_{t+1}$, then market participants assume that the investor followed the equilibrium strategy, but erred when pricing the order. The professional liquidity provider then updates his expectation about δ_t to the equilibrium value and posts a buy limit order at bid^*_{t+1}. The original investor's limit order then executes with zero probability.

If a limit buy order is posted at a price $\widehat{\mathsf{bid}} > \mathsf{bid}^*_{t+1}$, then participants believe that this order stems from an investor with a sufficiently high valuation (e.g., $z_t = 1$) and update their expectations about δ_t to $\mathsf{E}[\delta_t \mid \widehat{\mathsf{bid}}]$ accordingly.The new posterior expectation of V_t equals to $p_{t-1} + \mathsf{E}[\delta_t \mid \widehat{\mathsf{bid}}]$. The professional liquidity provider is then willing to post a bid price $\mathsf{bid}^{**}_{t+1} \leq p_{t-1} + \mathsf{E}[\delta_t \mid \widehat{\mathsf{bid}}] + \mathsf{E}[\delta_{t+1} \mid \mathsf{MS}_{t+1}]$. With the out-of-the-equilibrium belief of $\delta_t = 1$ and with the bid-ask spread< 1, a limit order with the new price bid^{**}_{t+1} outbids any limit buy order that yields investors positive expected profits.

The beliefs for an out-of-equilibrium sell order are symmetric. These out-of-equilibrium beliefs ensure that no investor deviates from his equilibrium strategy. I emphasize that these beliefs and actions do *not* materialize in equilibrium. Instead, they can be thought of as a "threat" to ensure that investors do not deviate from their prescribed equilibrium strategies.

Figure 5: Graphical Proofs

The panels below depict three plots in $(\mu, \rho) \in (0,1)^2$ that serve to show that the referenced equations are above zero for all μ and ρ. From top to bottom, the figures correspond to equations (87), (96), and (102).

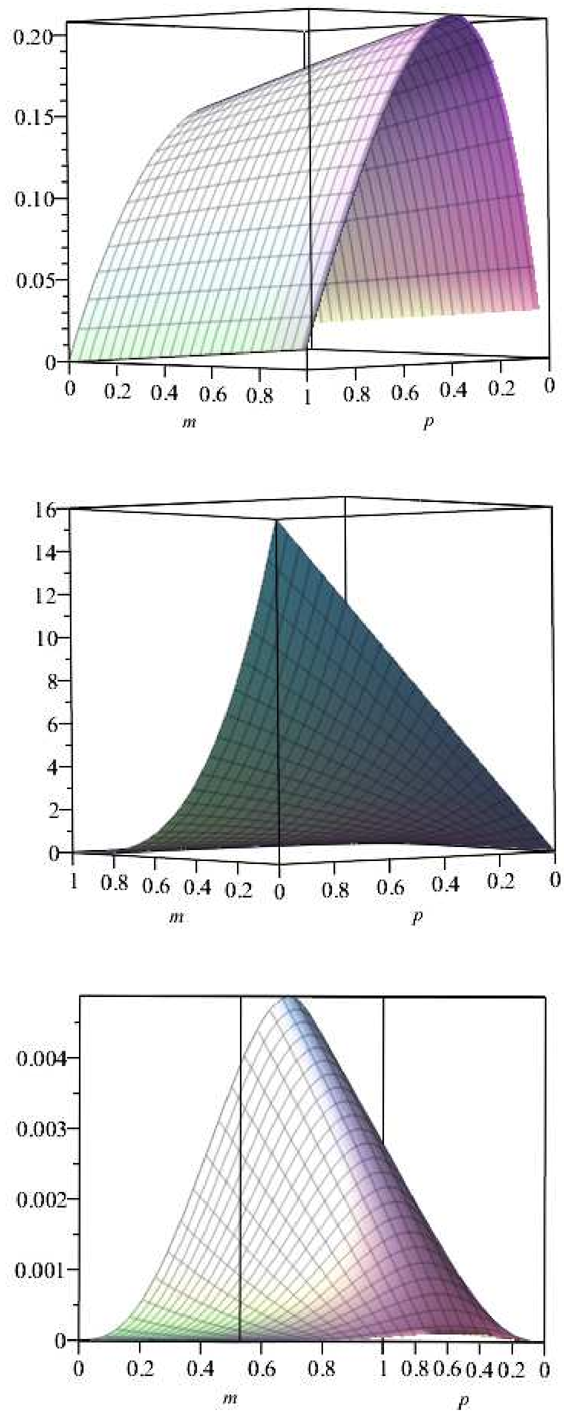

Figure 6: Volume, Market Participation

The panels below depict volume (lit market and total), and market participation as a function of the price improvement ($\lambda < \lambda^*$ on the left, $\lambda \geq \lambda^*$ on the right). Vertical dashed lines mark values for λ_1, λ_2, λ^*, λ_3 and λ_4; the horizontal dashed line indicates the lit market only benchmark value. Parameter $\mu = 0.5$ and $\rho = 0.95$. Results for other values of μ and ρ are qualitatively similar.

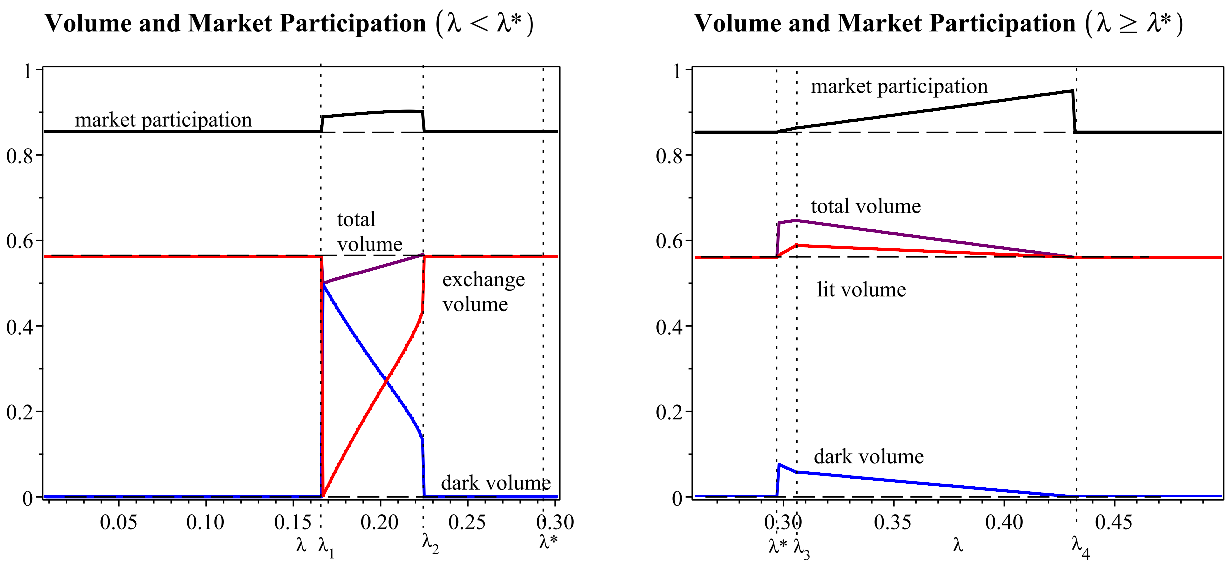

Figure 7: Price Efficiency

The panels below depict price efficiency as a function of the price improvement ($\lambda < \lambda^*$ on the left, $\lambda \geq \lambda_4$ on the right). Conditional price impact (blue line) indicates the expected price impact of an investor entering at period t conditional on a trade occurring; unconditional price impact (red line) describes the expected price impact of an investor entering the market at period t. The horizontal dashed line indicates the lit market only benchmark value; higher values than the benchmark are less efficient. Vertical dashed lines mark values for λ_1, λ_2, λ^*, λ_3 and λ_4. Parameters $\mu = 0.5$ and $\rho = 0.95$. Results for other values of μ and ρ are qualitatively similar.